Transitivity degrees of countable groups and acylindrical hyperbolicity

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Abstract

We prove that every countable acylindrically hyperbolic group admits a highly transitive action with finite kernel. This theorem uniformly generalizes many previously known results and allows us to answer a question of Garion and Glassner on the existence of highly transitive faithful actions of mapping class groups. It also implies that in various geometric and algebraic settings, the transitivity degree of an infinite group can only take two values, namely 1 and ∞. Here by *transitivity degree* of a group we mean the supremum of transitivity degrees of its faithful permutation representations. Further, for any countable group G admitting a highly transitive faithful action, we prove the following dichotomy: Either G contains a normal subgroup isomorphic to the infinite alternating group or G resembles a free product from the model theoretic point of view. We apply this theorem to obtain new results about universal theory and mixed identities of acylindrically hyperbolic groups. Finally, we discuss some open problems.

Contents

1 Introduction

Recall that an action of a group G on a set Ω is *k-transitive* if $|\Omega| \geq k$ and for any two k-tuples of distinct elements of Ω, $(a_1, ..., a_k)$ and $(b_1, ..., b_k)$, there exists $g \in G$ such that $ga_i = b_i$ for $i = 1, ..., k$. The *transitivity degree* of a countable group G, denoted $\mathrm{td}(G)$, is the supremum of all $k \in \mathbb{N}$ such that G admits a k-transitive faithful action. For finite groups, this notion is classical and fairly well understood. It is easy to see that $\mathrm{td}(S_n) = n$, $\mathrm{td}(A_n) = n - 2$, and it is a consequence of the classification of finite simple groups that any finite group G other than S_n or A_n has $\mathrm{td}(G) \leq 5$. Moreover, if G is not S_n, A_n, or one of the Mathieu groups M_{11}, M_{12}, M_{23}, M_{24}, then $td(G) \leq 3$ (see [18]).

For infinite groups, however, very little is known. For example, we do not know the answer to the following basic question: Does there exist an infinite countable group of transitivity degree k for every $k \in \mathbb{N}$? There are examples for $k = 1, 2, 3$, and ∞, but the problem seems open even for $k = 4$. There is also a new phenomenon, which does not occur in the finite world: highly transitive actions. Recall that an action of a group is *highly transitive* if it is k-transitive for all $k \in \mathbb{N}$. We say that a group is *highly transitive* if it admits a highly transitive faithful action; it is easy to see that a countably infinite group is highly transitive if and only if it embeds as a dense subgroup in the infinite symmetric group $Sym(\mathbb{N})$ endowed with the topology of pointwise convergence. Obviously $\mathrm{td}(G) = \infty$ whenever G is highly transitive, but we do not know if the converse is true. Yet another interesting question is whether there exists a reasonable classification of highly transitive groups (or, more generally, groups of high transitivity degree). The main goal of this paper is to address these questions in certain geometric and algebraic settings.

We begin by discussing known examples of highly transitive groups. Trivial examples of such groups are $FSym(\mathbb{N})$ (the group of all finitary permutations of $\mathbb{N}$) and the infinite alternating group $Alt(\mathbb{N})$ (the group of all finitary permutations that act as even permutations on their supports). Clearly every subgroup of $Sym(\mathbb{N})$ that contains $Alt(\mathbb{N})$ is also highly transitive. A fairly elementary argument allows one to construct finitely generated groups of this sort; we record the following.

Proposition 1.1 (Prop. 5.13). *For every finitely generated infinite group Q, there exists a finitely generated group G such that $FSym(\mathbb{N}) \triangleleft G \leq Sym(\mathbb{N})$ and $G/FSym(\mathbb{N}) \cong Q$. In particular, G is highly transitive.*

This proposition can be used, for instance, to construct finitely generated highly transitive torsion groups by taking Q to be finitely generated and torsion. Another interesting particular example of this sort is the family of Houghton groups H_n, $n \geq 2$, which correspond to the case $Q = \mathbb{Z}^{n-1}$. These groups are elementary amenable and finitely presented for $n \geq 3$ (see Example 6.6).

On the other hand, there are highly transitive groups of a completely different nature. The first such an example is F_n, the free group of rank $n \geq 2$. Highly transitive faithful actions of F_n were constructed by McDonough [39], and it was shown by Dixon [17] that

in some sense most finitely generated subgroups of $Sym(\mathbb{N})$ are free and highly transitive. Other known examples of highly transitive groups include fundamental groups of closed surfaces of genus at least 2 [35] and, more generally, non-elementary hyperbolic groups without non-trivial finite normal subgroups [14], all free products of non-trivial groups except $\mathbb{Z}_2 * \mathbb{Z}_2$ [22, 26, 28, 42] and many other groups acting on trees [20]. Garion and Glassner [21] used an interesting approach employing Tarski Monster groups constructed by Olshanskii [43] to show that $Out(F_n)$ is highly transitive for $n \geq 4$. Motivated by similarity between $Out(F_n)$ and mapping class groups, they ask which mapping class groups are highly transitive. Another question from [21] is whether $Out(F_3)$ is highly transitive.

The first main result of our paper is a theorem which simultaneously generalizes most results mentioned in the previous paragraph and allows us to answer the questions of Garion and Glassner. Recall that an isometric action of a group G on a metric space (X, d) is *acylindrical* if for all $\varepsilon > 0$, there exist $R > 0, N > 0$ such that for all $x, y \in X$ with $d(x, y) \geq R$, there are at most N elements $g \in G$ satisfying $d(x, gx) \leq \varepsilon$ and $d(y, gy) \leq \varepsilon$. For example, it is easy to show that the action of a fundamental group of a graph of groups with finite edge groups on the associated Bass-Serre tree is acylindrical. A group is called *acylindrically hyperbolic* if it is not virtually cyclic and admits an acylindrical action on a hyperbolic metric space with unbounded orbits [49]. The class of acylindrically hyperbolic groups includes many examples of interest; for an extensive list we refer to [15, 41, 49, 48].

Every acylindrically hyperbolic group G has a maximal normal finite subgroup called the *finite radical* of G and denoted $K(G)$ [15].

Theorem 1.2. *Every countable acylindrically hyperbolic group admits a highly transitive action with finite kernel. In particular, every countable acylindrically hyperbolic group with trivial finite radical is highly transitive.*

It is well-known and easy to see that a group of transitivity degree at least 2 cannot have finite normal subgroups (see Lemma 4.2 (b)). Thus the kernel of a highly transitive action of a countable acylindrically hyperbolic group G cannot be smaller than $K(G)$. In particular, the assumption about triviality of finite radical cannot be dropped from the second sentence of the theorem.

Our proof of Theorem 1.2 is based on the notion of a small subgroup of an acylindrically hyperbolic group introduced in Section 2. This notion seems to be of independent interest. In addition, the proof of Theorem 1.2 makes use of hyperbolically embedded subgroups introduced in [15] and small cancellation theory in acylindrically hyperbolic groups developed in [32].

Here we mention just few particular cases of our theorem considered in Corollaries 4.9, 4.14, and 4.19–4.23. By $\Sigma_{g,n,p}$ we denote a p times punctured compact orientable surface of genus g with n boundary components.

Corollary 1.3. *(a) Let G be a countable group hyperbolic relative to a collection of proper subgroups. Then G is highly transitive if and only if it is not virtually cyclic and has no non-trivial finite normal subgroups.*

(b) *For $g, n, p \in \mathbb{N} \cup \{0\}$, the mapping class group $\mathrm{Mod}(\Sigma_{g,n,p})$ is highly transitive if and only if $n = 0$, $3g + p \geq 5$, and $(g, p) \neq (2, 0)$.*

(c) *(cf. [21, Theorem 1]) $\mathrm{Out}(F_n)$ is highly transitive if and only if $n \geq 3$.*

(d) *Let M be a compact irreducible 3-manifold. Then $\pi_1(M)$ is highly transitive if and only if it is not virtually solvable and M is not Seifert fibered.*

(e) *A right angled Artin group is highly transitive if and only if it is non-cyclic and directly indecomposable.*

(f) *Every 1-relator group with at least 3 generators is highly transitive.*

Recall that a free product of two non-trivial groups has no non-trivial finite normal subgroups. Thus part (a) of Corollary 1.3 covers the case of free products considered in [22, 26, 28] as well as some results of [20]; it also covers the result about hyperbolic groups [14]. Parts (b) and (c) answer the questions of Garion and Glassner mentioned above. Our contribution to (c) is the case $n = 3$.

Theorem 1.2 also allows us to show that in various geometric and algebraic settings, transitivity degree of an infinite group can only take two values, namely 1 and ∞, and infinite transitivity degree is equivalent to being highly transitive. More precisely, we consider three conditions for a group G:

(C$_1$) $\mathrm{td}(G) = \infty$.

(C$_2$) G is highly transitive.

(C$_3$) G is acylindrically hyperbolic with trivial finite radical.

In Section 4, we show that an infinite subgroup G of a mapping class group is either acylindrically hyperbolic or satisfies certain algebraic conditions which imply $\mathrm{td}(G) = 1$. A similar result for subgroups of 3-manifold groups was obtained in [41]. Combining this with Theorem 1.2, we obtain the following.

Corollary 1.4 (Cor. 4.13, Cor. 4.15)**.** *Let G be an infinite subgroup of $\mathrm{Mod}(\Sigma_{g,n,p})$ for some $g, n, p \in \mathbb{N} \cup \{0\}$ or an infinite subgroup of the fundamental group of a compact 3-manifold. Then $\mathrm{td}(G) \in \{1, \infty\}$ and conditions (C$_1$)–(C$_3$) are equivalent.*

A similar result can also be proved in some algebraic settings, e.g., for subgroups of finite graph products. Let Γ be a graph (without loops or multiple edges) with vertex set V and let $\{G_v\}_{v \in V}$ be a family of groups indexed by vertices of Γ. The *graph product* of $\{G_v\}_{v \in V}$ with respect to Γ, denoted $\Gamma\{G_v\}_{v \in V}$, is the quotient group of the free product $*_{v \in V} G_v$ by the relations $[g, h] = 1$ for all $g \in G_u$, $h \in G_v$ whenever u and v are adjacent in Γ. Graph products simultaneously generalize free and direct products of groups. Basic

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examples are right angled Artin and Coxeter groups, which are graph products of copies of $\mathbb{Z}$ and $\mathbb{Z}_2$, respectively. The study of graph products and their subgroups has gained additional importance in view of the recent breakthrough results of Agol, Haglund, Wise, and their co-authors, which show that many groups can be virtually embedded into right angled Artin groups (see [1, 27, 57] and references therein).

Corollary 1.5 (Cor. 4.17). *Let G be a countably infinite subgroup of a finite graph product $\Gamma\{G_v\}_{v \in V}$. Suppose that G is not isomorphic to a subgroup of one of the multiples. Then $\mathrm{td}(G) \in \{1, 2, \infty\}$ and conditions (C_1)–(C_3) are equivalent. If, in addition, every G_v is residually finite, then $\mathrm{td}(G) \in \{1, \infty\}$.*

To state our next result we need some preparation. Let F_n denote the free group of rank n and recall that a group G satisfies a *mixed identity* $w = 1$ for some $w \in G * F_n$ if every homomorphism $G * F_n \to G$ that is identical on G sends w to 1. We say that the mixed identity $w = 1$ is non-trivial if $w \neq 1$ as an element of $G * F_n$. For the general theory of mixed identities and mixed varieties of groups we refer to [3]. We say that G is *mixed identity free* (or *MIF* for brevity) if it does not satisfy any non-trivial mixed identity.

The property of being MIF is much stronger than being identity free and imposes strong restrictions on the algebraic structure of G. For example, if G has a non-trivial center, then it satisfies the non-trivial mixed identity $[a, x] = 1$, where $a \in Z(G) \setminus \{1\}$. Similarly, it is easy to show that an MIF group has no finite normal subgroups, is directly indecomposable, has infinite girth, etc. (see Proposition 5.4). Other examples of groups satisfying a non-trivial mixed identity are Thompson's group F [55] or any subgroup of $Sym(\mathbb{N})$ containing $Alt(\mathbb{N})$ (see Theorem 1.6 below).

It is also worth noting that MIF groups resemble free products from the model theoretic point of view. More precisely, a countable group G is MIF if and only if G and $G * F_n$ are universally equivalent as G-groups for all $n \in \mathbb{N}$. This means that a universal first order sentence with constants from G holds true in G if and only if it holds true in $G * F_n$. Groups universally equivalent to G as G-groups are exactly the coordinate groups of irreducible algebraic varieties over G [7]. For more details we refer to Section 5 and Proposition 5.3.

In Section 5, we prove the following dichotomy for highly transitive countable groups, which captures the principal difference between examples provided by Proposition 1.1 and Theorem 1.2.

Theorem 1.6. *Let G be a highly transitive countable group. Then exactly one of the following two mutually exclusive conditions holds.*

(a) *G contains a normal subgroup isomorphic to $Alt(\mathbb{N})$.*

(b) *G is MIF (equivalently, G and $G * F_n$ are universally equivalent as G-groups for every $n \in \mathbb{N}$).*

It is known that infinite normal subgroups of acylindrically hyperbolic groups are also acylindrically hyperbolic and hence cannot be torsion [49]. Thus condition (a) from Theorem
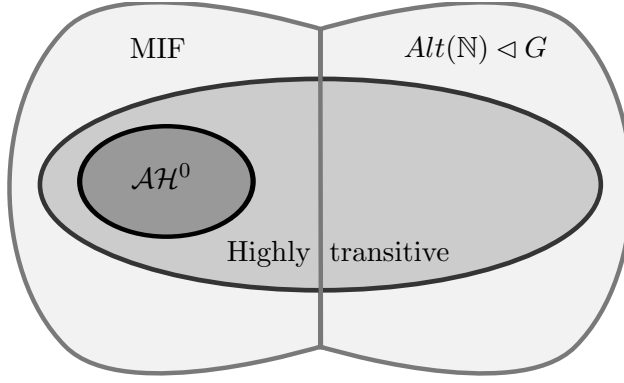
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Figure 1:

1.6 cannot hold if G is acylindrically hyperbolic. Combining this with Theorem 1.2 we obtain the following.

Corollary 1.7 (Cor. 5.10). *Let G be an acylindrically hyperbolic group with trivial finite radical. Then G is MIF.*

Clearly G is not MIF whenever $K(G) \neq \{1\}$. In the particular case of non-cyclic torsion free hyperbolic groups, the above corollary was proved in [2] by different methods.

Theorem 1.2 and Theorem 1.6 are summarized in Fig. 1. Here by $\mathcal{AH}^0$ we denote the class of acylindrically hyperbolic groups with trivial finite radical. In Section 5 we show that all inclusions are proper. We mention two examples.

Proposition 1.8 (Cor. 5.12, Cor. 5.17). *(a) The group $G = \langle a, b, t \mid a^b = a^2, [a, t] = 1 \rangle$ is highly transitive and MIF, but not acylindrically hyperbolic.*

(b) There exist finitely generated MIF groups of transitivity degree 1.

The paper is organized as follows: In Section 2 we review properties of acylindrically hyperbolic groups and hyperbolically embedded subgroups necessary for the proof of Theorem 1.2. In Section 3 we prove Theorem 1.2, and in Section 4 we apply this theorem to various classes of groups to obtain Corollaries 1.3-1.5. In Section 5 we study the relationship between mixed identities and highly transitive actions and prove Theorem 1.6. Some open questions and relevant examples are discussed in Section 6.

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2   Preliminaries on acylindrically hyperbolic groups

2.1. Generating alphabets and Cayley graphs.   When dealing with relative presentations of groups, we often need to represent the same element of a group G by several distinct generators. Thus, instead of a generating set of G, it is more convenient to work with an alphabet $\mathcal{A}$ given together with a (not necessarily injective) map $\alpha\colon \mathcal{A} \to G$ such that $\alpha(\mathcal{A})$ generates G. We begin by formalizing this approach.

Let $\mathcal{A}$ be a set, which we refer to as an *alphabet*. Let G be a group and let $\alpha\colon \mathcal{A} \to G$ be a (not necessarily injective) map. We say that G is *generated* by $\mathcal{A}$ (or $\mathcal{A}$ is a *generating alphabet* of G) if G is generated by $\alpha(\mathcal{A})$. Note that a generating set $X \subseteq G$ can be considered as a generating alphabet with the obvious map $\alpha\colon X \to G$.

By the *Cayley graph* of G with respect to a generating alphabet $\mathcal{A}$, denoted $\Gamma(G, \mathcal{A})$, we mean a graph with vertex set G and the set of edges defined as follows. For every $a \in \mathcal{A}$ and every $g \in G$, there is an oriented edge $(g, g\alpha(a))$ in $\Gamma(G, \mathcal{A})$ labelled by a. Given a (combinatorial) path p in $\Gamma(G, \mathcal{A})$, we denote by $\mathbf{Lab}(p)$ its label. Note that if α is not injective, $\Gamma(G, \mathcal{A})$ may have multiple edges. Of course, the identity map on G induces an isometry between vertex sets of the graphs $\Gamma(G, \mathcal{A})$ and $\Gamma(G, \alpha(\mathcal{A}))$. Thus we only need to distinguish between $\Gamma(G, \mathcal{A})$ and $\Gamma(G, \alpha(\mathcal{A}))$ when dealing with labels; in all purely metric considerations we do not need generating alphabets and can simply work with generating sets.

Given a generating set X of G and an element $g \in G$, let $|g|_X$ denote the word length of g with respect to X, that is the length of a shortest word in $X^{\pm 1}$ that represents g. For $g, h \in G$, we define $d_X(g, h) = |g^{-1}h|_X$. Finally, let

$$B_X(n) = \{g \in G \mid |g|_X \le n\}.$$

2.2. Hyperbolically embedded subgroups.   The typical situation when we apply the language described above is the following. Suppose that we have a group G, a subgroup H of G, and a subset $X \subseteq G$ such that X and H together generate G. We think of X and H as abstract sets and consider the disjoint union

$$\mathcal{A} = X \sqcup H, \tag{1}$$

and the map $\alpha\colon \mathcal{A} \to G$ induced by the obvious maps $X \to G$ and $H \to G$. By abuse of notation, we do not distinguish bretween subsets X and H of G and their preimages in $\mathcal{A}$. This will not create any problems since the restrictions of α on X and H are injective. Note, however, that α is not necessarily injective. Indeed if X and subgroup H intersect in G, then every element of $H \cap X \subseteq G$ will have at least two preimages in $\mathcal{A}$: one in X and another in H (since we use disjoint union in (1)).

Convention 2.1. Henceforth we always assume that generating sets and relative generating sets are symmetric. That is, if $x \in X$, then $x^{-1} \in X$. In particular, every element of G can be represented by a word in $X \sqcup H$.

In these settings, we consider the Cayley graphs $\Gamma(G, X \sqcup H)$ and $\Gamma(H, H)$ and naturally think of the latter as a subgraph of the former. We introduce a *relative metric* $\widehat{\mathrm{d}} \colon H \times H \to [0, +\infty]$ as follows. We say that a path p in $\Gamma(G, X \sqcup H)$ is *admissible* if it contains no edges of $\Gamma(H, H)$. Let $\widehat{\mathrm{d}}(h, k)$ be the length of a shortest admissible path in $\Gamma(G, X \sqcup H)$ that connects h to k. If no such a path exists, we set $\widehat{\mathrm{d}}(h, k) = \infty$. Clearly $\widehat{\mathrm{d}}$ satisfies the triangle inequality, where addition is extended to $[0, +\infty]$ in the natural way.

Definition 2.2. A subgroup H of G is *hyperbolically embedded in G with respect to a subset* $X \subseteq G$, denoted $H \hookrightarrow_h (G, X)$, if the following conditions hold.

(a) The group G is generated by X together with H and the Cayley graph $\Gamma(G, X \sqcup H)$ is hyperbolic.

(b) Any ball (of finite radius) in H with respect to the metric $\widehat{\mathrm{d}}$ contains finitely many elements.

Further we say that H is *hyperbolically embedded* in G and write $H \hookrightarrow_h G$ if $H \hookrightarrow_h (G, X)$ for some $X \subseteq G$.

Hyperbolically embedded subgroups were introduced and studied in [15] as a generalizaton of relatively hyperbolic groups; indeed, G is hyperbolic relative to H if and only if $H \hookrightarrow_h (G, X)$ with $|X| < \infty$ [15, Proposition 4.28]. The following lemma is a particular case of this result.

Lemma 2.3. *For any group G, $G \hookrightarrow_h (G * \langle t \rangle, \{t\})$.*

In the next two lemmas, we let G be a group and H a subgroup of G.

The following is a particular case of [15, Proposition 4.35].

Lemma 2.4. *Suppose that $H \hookrightarrow_h (G, X)$ for some $X \subseteq G$ and there is a subset $Y \subseteq H$ and a subgroup $K \hookrightarrow_h (H, Y)$. Then $K \hookrightarrow_h (G, X \cup Y)$.*

Given a group G, a subgroup $H \leq G$, and $g \in G$, we denote by H^g the conjugate $g^{-1} H g$.

Lemma 2.5. *[15, Proposition 2.8] If $H \hookrightarrow_h G$, then for all $g \in G \setminus H$, $|H \cap H^g| < \infty$.*

2.3. Acylindrically hyperbolic groups. Recall that an isometric action of a group G on a metric space S is *acylindrical* if for every $\varepsilon > 0$, there exist $R, N > 0$ such that for every two points x, y with $\mathrm{d}(x, y) \geq R$, there are at most N elements $g \in G$ satisfying

$$\mathrm{d}(x, gx) \leq \varepsilon \quad \text{and} \quad \mathrm{d}(y, gy) \leq \varepsilon.$$

Given a group G acting on a hyperbolic space S, an element $g \in G$ is called *loxodromic* if the map $\mathbb{Z} \to S$ defined by $n \mapsto g^n s$ is a quasi-isometry for some (equivalently, any) $s \in S$.

Every loxodromic element $g \in G$ has exactly 2 limit points $g^{\pm\infty}$ on the Gromov boundary ∂S. Loxodromic elements $g, h \in G$ are called *independent* if the sets $\{g^{\pm\infty}\}$ and $\{h^{\pm\infty}\}$ are disjoint.

We will often use the following.

Theorem 2.6 ([49, Theorem 1.1])**.** *Let G be a group acting acylindrically on a hyperbolic space. Then exactly one of the following three conditions holds.*

(a) G is elliptic.

(b) G contains a loxodromic element g such that $|G : \langle g \rangle| < \infty$.

(c) G contains infinitely many loxodromic elements that are independent (i.e., have disjoint limit sets on the boundary).

Definition 2.7. A group G is called *acylindrically hyperbolic* if it admits a non-elementary acylindrical action on a hyperbolic space.

In the case of acylindrical actions on hyperbolic spaces being non-elementary is equivalent to the action having unbounded orbits and G being not virtually cyclic by theorem 2.6.

It is easy to see from the definition that for every group G we have $G \hookrightarrow_h G$ and $K \hookrightarrow_h G$ for every finite subgroup K of G. Following [15], we call a hyperbolically embedded subgroup $H \hookrightarrow_h G$ *non-degenerate* if H is infinite and $H \neq G$.

The next result is a part of [49, Theorem 2.2]. In particular, it allows us to apply all results from [15] concerning groups with non-degenerate hyperbolically embedded subgroups to acylindrically hyperbolic groups.

Theorem 2.8. *A group G is acylindrically hyperbolic if and only if G contains a non-degenerate hyperbolically embedded subgroup.*

If G is acylindrically hyperbolic, then G has a unique, maximal finite normal subgroup $K(G)$ called the *finite radical* of G [15, Theorem 6.14].

Lemma 2.9. *[32] Let G be acylindrically hyperbolic. Then $G/K(G)$ is acylindrically hyperbolic and has trivial finite radical.*

2.4. Small subgroups in acylindrically hyperbolic groups. Given a group G and a generating set (or an alphabet) $\mathcal{A}$, we say that the Cayley graph $\Gamma(G, \mathcal{A})$ is *acylindrical* if so is the action of G on $\Gamma(G, \mathcal{A})$. A hyperbolic space S is called *non-elementary* if $|\partial S| > 2$, where ∂S denotes the Gromov boundary of S.

Definition 2.10. We call a subgroup H of a group G *small in G* (or just *small* if G is understood) if there exists a generating set X of G such that $H \subseteq X$ and $\Gamma(G, X)$ is hyperbolic, non-elementary, and acylindrical.

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Note that every group containing small subgroups is acylindrically hyperbolic.

Lemma 2.11. *Assume that a group G is acylindrically hyperbolic and $K(G) = \{1\}$. Then for every small subgroup $H \leq G$, the action of G on the coset space G/H is faithful.*

Proof. Let N denote the kernel of the action of G on G/H. Let X be a generating set of G such that $H \subseteq X$ and $\Gamma(G, X)$ is acylindrical, hyperbolic, and non-elementary. Note that N acts elliptically on $\Gamma(G, X)$ as $N \leq H \subseteq X$. Recall that for every acylindrical non-elementary action of a group on a hyperbolic space, every normal elliptic subgroup is finite [49, Lemma 7.2]. Applying this to the group N acting on $\Gamma(G, X)$, we obtain that $N \leq K(G)$ and consequently $N = \{1\}$. □

The proof of Theorem 1.2 will make use of one particular example of small subgroups. Recall that given a generating set X of a group G, $|\cdot|_X$ and d_X denote the corresponding word length and metric on G, respectively.

Proposition 2.12. *Let G be a group, F a subgroup of G, X a subset of G such that $F \hookrightarrow_h (G, X)$. Suppose that a subgroup $H \leq G$ is generated by a subset Z such that*

$$\sup_{z \in Z} |z|_X < \infty$$

and

$$\mathrm{d}_Z(h_1, h_2) \leq K \mathrm{d}_{X \cup F}(h_1, h_2)$$

for all $h_1, h_2 \in H$. Then H is small. Moreover, for every subgroup $S \leq G$ that properly contains F, there exists a subset $Y \subseteq G$ such that $\Gamma(G, Y)$ is hyperbolic and acylindrical, $H \subseteq Y$, and the action of S on $\Gamma(G, Y)$ is non-elementary.

The proof can be easily extracted from [49]. In order to make our paper self-contained, we recall necessary definitions and results from [49].

Let $\mathcal{A} = X \sqcup F$ and let p be a (combinatorial) path in $\Gamma(G, \mathcal{A})$. A non-trivial subpath q of p is called an *F-component* if **Lab**(q) is a word in the alphabet F and q is not contained in any bigger subpath of p with this property. If q is an F-component of p and a, b are endpoints of q, then all vertices of q belong to the same coset $aF = bF$; in this case we say that p *penetrates* aF and call the number $\widehat{\mathrm{d}}(1, a^{-1}b)$ the *depth of the penetration*. Given two elements $f, g \in G$, we say that a coset aF is $(f, g; D)$-*separating* if there exists a geodesic p from f to g in $\Gamma(G, \mathcal{A})$ such that p penetrates aF and the depth of the penetration is greater than D.

The term "separating" is justified by the following result, which is a simplification of [49, Lemma 4.5].

Lemma 2.13. *Let G be a group, F a subgroup of G, X a subset of G such that $F \hookrightarrow_h (G, X)$. Then for any constants μ and b, there exists $C = C(\mu, b)$ such that the following holds. Let $f, g \in G$ and let aF be an $(f, g; C)$-separating coset. Then every (μ, b)-quasi-geodesic in $\Gamma(G, X \sqcup F)$ that connects f to g penetrates aF.*

It is observed in [49] that if $F \hookrightarrow_h (G, X)$, then the Cayley graph $\Gamma(G, X \sqcup H)$ is not necessarily acylindrical. However, the following lemma holds true. Its first claim is a particular case of Theorem 5.4 in [49]; the second claim is not stated in that theorem itself, but is obvious from its proof.

Lemma 2.14. *Let G be a group, F a subgroup of G, X a subset of G such that $F \hookrightarrow_h (G, X)$. Then there exists a set $Y \supseteq X$ such that $F \hookrightarrow_h (G, Y)$ and $\Gamma(G, Y \sqcup F)$ is acylindrical. More precisely, given a constant $D > 0$, let $Y = Y(D)$ denote the set of all elements $y \in G$ such that the set of all $(1, y; D)$-separating cosets is empty. Then for every large enough D, the set Y satisfies the above requirements.*

We are now ready to prove Proposition 2.12.

Proof. Let $M = \sup_{z \in Z} |z|_X$. For every element $h \in H$, let p_h be a shortest path in $\Gamma(G, X \sqcap H)$ connecting 1 to h such that $\mathbf{Lab}(p_h)$ is a word in the alphabet X. In particular, p_h does not penetrate any left coset of F. It is straightforward to check that p_h is (μ, b)-quasi-geodesic for some μ and b that only depend on K and M (in fact, we can take $\mu = KM$ and $b = 2KM^2$).

Let D be a constant such that the conclusion of Lemma 2.14 holds and $D \geq C(\mu, b)$, where $C(\mu, b)$ is given by Lemma 2.13. Then for any $h \in H$, there are no $(1, h; D)$-separating cosets. Therefore, the set $Y = Y_D$ defined in Lemma 2.14 contains H. By Lemma 2.14, $\Gamma(G, Y \sqcup F)$ is acylindrical.

It remains to show that the action of S on $\Gamma(G, Y \sqcup F)$ is non-elementary. By Theorem 2.6, it suffices to show that S has unbounded orbits and is not virtually cyclic. Let $a \in S \setminus F$. By [15, Corollary 6.12], there exists $f \in F$ such that af is loxodromic with respect to the action on $\Gamma(G, Y \sqcup F)$. Since $af \in S$, we obtain that the orbits of S are unbounded. Finally assume that S is elementary. Since F is infinite, it must be of finite index in S. In particular, F contains an infinite subgroup N such that $N \triangleleft S$. Then for every $a \in S$, we obtain $|F^a \cap F| \geq |N| = \infty$ which contradicts Lemma 2.5 if $a \notin F$. Thus the action of S on $\Gamma(G, Y \sqcup F)$ is non-elementary. $\qquad\square$

3 Highly transitive actions of acylindrically hyperbolic groups

3.1. Outline of the proof. The goal of this section is to prove Theorem 1.2. We begin with a brief sketch of the proof.

We first note that by Lemma 2.9, it suffices to deal with the case of trivial finite radical. Assuming $K(G) = \{1\}$, we will construct a subgroup $H_\infty \leq G$ such that the action of G on the left cosets of H_∞ will be highly transitive. H_∞ will be defined as the union of a sequence of subgroups

$$H_1 \leq H_2 \leq \dots$$

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Given H_i and two tuples of pairwise distinct cosets $(a_1 H_i, ..., a_k H_i)$ and $(b_1 H_i, ..., b_k H_i)$, we define H_{i+1} by choosing an element $g_i \in G$ and setting

$$H_{i+1} = \langle H_i, b_1^{-1} g_i a_1, ..., b_k^{-1} g_i a_k \rangle. \tag{2}$$

Then clearly $g_i a_j H_{i+1} = b_j H_{i+1}$, so after enumerating all tuples of elements of G and repeating this process inductively we will get that the action of G on the set Ω of left cosets of H_∞ is highly transitive, provided $|\Omega| = \infty$. To the best of our knowledge, this natural construction first appeared in [4]. It was also used by Chaynikov [14] to prove Theorem 1.2 in the particular case when G is hyperbolic.

The main difficulty is to ensure that that the action of G on Ω is faithful. Note that if the elements g_i is not chosen carefully, then we are likely get $H_\infty = G$. In Chainikov's paper [14], the proof of faithfulness of the action is built on the well-developed theory of quasi-convex subgroups of hyperbolic groups. Currently there is no analogue of this theory in the more general context of acylindrically hyperbolic groups, so we choose another approach based on the notion of a small subgroup introduced in the previous section and small cancellation theory in acylindrically hyperbolic groups developed in [32].

More precisely, we prove in Lemma 3.1 that if H_i is small for some i, then by choosing the element g_i carefully (small cancellation theory is employed here) one can ensure that the subgroup H_{i+1} defined by (2) is still small. In addition, we show that is B is any finite subset of G disjoint from H_i, we can keep it disjoint from H_{i+1}. Recall also that the action of G on the space of left cosets G/H is faithful whenever H is a small subgroup and $K(G) = \{1\}$ (Lemma 2.11). These results allow us to iterate the above construction in such a way that the resulting action of G on G/H_∞ is faithful.

3.2. The inductive step. The main goal of this subsection is to prove the following proposition, which takes care of the inductive step in the proof of Theorem 1.2. Given a group G, a subgroup H, and two collections of elements $\bar{a} = (a_1, ..., a_k)$ and $\bar{b} = (b_1, ..., b_k)$ of G, we say that the triple $(\bar{a}, \bar{b}, H)$ is *admissible* if

$$a_i H \neq a_j H \quad \text{and} \quad b_i H \neq b_j H \tag{3}$$

for all $1 \leq i < j \leq k$.

Proposition 3.1. *Let G be a group, H a small subgroup of G, $\bar{a} = (a_1, \ldots, a_k)$ and $\bar{b} = (b_1, \ldots, b_k)$ elements of G^k. Assume that the triple $(\bar{a}, \bar{b}, H)$ is admissible. Let also $B \subseteq G$ be a finite subset disjoint from H. Then there exists $g \in G$ and a subgroup $K \leq G$ such that*

(a) $H \leq K$.

(b) K is small in G.

(c) $g a_i K = b_i K$ for all $1 \leq i \leq k$.

(d) $K \cap B = \emptyset$.

We first recall some well-known results and terminology concerning free products.

Theorem 3.2. *[36, Chapt IV, Theorem 1.2] Let G_1, G_2 be arbitrary groups. Each element of a free product $G_1 * G_2$ can be uniquely expressed as $g_1 g_2 ... g_n$ where $n \geq 0$, each g_i is a non-trivial element of one of the factors and successive elements $g_i g_{i+1}$ belong to different factors.*

The expression $g_1 g_2 ... g_n$ as in the above theorem representing an element $g \in G_1 * G_2$ is called the *normal form* of the element g. The following corollary is an immediate consequence of the uniqueness of normal forms.

Corollary 3.3. *Let $g \in G_1 * G_2$ and let S_i be a generating set for G_i. Let $g_1 g_2 ... g_n$ be the normal form for g, where $g_i \in G_{j_i}$. Then*

$$|g|_{S_1 \cup S_2} = \sum_{i=1}^{n} |g_i|_{S_{j_i}}.$$

In particular, the embeddings of metric spaces $(G_i, d_{S_i}) \hookrightarrow (G_1 * G_2, d_{S_1 \cup S_2})$ are isometric.

Lemma 3.4. *Let G be a group with a generating set A, H a subgroup of G such that $H \subseteq A$. Let $\bar{a} = (a_1, ..., a_k)$, $\bar{b} = (b_1, ..., b_k)$ be elements of G^k. Assume that the triple $(\bar{a}, \bar{b}, H)$ is admissible and let $P = G * \langle t \rangle$,*

$$L = \langle H, b_1^{-1} t a_1, ..., b_k^{-1} t a_k \rangle \leq P,$$

and

$$Z = H \cup \{ b_1^{-1} t a_1, ..., b_k^{-1} t a_k \} \cup \{ a_1^{-1} t^{-1} b_1, ..., a_k^{-1} t^{-1} b_k \}.$$

Then $L \cap G = H$ and the natural map $(L, d_Z) \to (P, d_{A \cup \{t\}})$ is Lipschitz.

Proof. First, if $h \in H \setminus \{1\}$, then $|h|_Z = |h|_{A \cup \{t\}} = 1$. Assume now that $h \in L \setminus H$. Let

$$r_i = b_i^{-1} t a_i$$

for $1 \leq i \leq k$ and let W be a minimal word in Z representing h. Then

$$W = h_1 r_{i_1}^{\varepsilon_1} ... h_m r_{i_m}^{\varepsilon_m} h_{m+1}$$

for some $m \geq 1$, where each h_j is a (possibly trivial) element of H and $\varepsilon_j \in \{\pm 1\}$. Then $|h|_Z = \|W\| = \sum_{i=1}^{m+1} |h_i|_Z + m$, and note that each $|h_i|_Z \in \{0, 1\}$. Let $g_j = u_j h_j v_j$, where

$$u_j = \begin{cases} a_{i_{j-1}} & \text{if } \varepsilon_{j-1} = 1 \\ b_{i_{j-1}} & \text{if } \varepsilon_{j-1} = -1 \end{cases}$$

for $2 \leq j \leq m+1$ and

$$v_j = \begin{cases} b_{i_j}^{-1} & \text{if } \varepsilon_j = 1 \\ a_{i_j}^{-1} & \text{if } \varepsilon_j = -1 \end{cases}$$

for $1 \leq j \leq m$. We also define $u_1 = v_{m+1} = 1$. Then

$$h = g_1 t^{\varepsilon_1}...g_m t^{\varepsilon_m} g_{m+1}. \tag{4}$$

We show now that consecutive t-letters in this expression do not cancel, which means that (4) becomes a normal form for h in the free product $G * \langle t \rangle$ after possibly removing the trivial g_j and combining adjacent powers of t. Suppose for some $2 \leq j \leq m$, $\varepsilon_{j-1} = 1$ and $\varepsilon_j = -1$. Then $g_j = a_{i_{j-1}} h_j a_{i_j}^{-1}$. If $i_{j-1} \neq i_j$, then $g_j \neq 1$ since $a_{i_{j-1}} H \neq a_{i_j} H$. If $i_{j-1} = i_j$, then $h_j \neq 1$, for otherwise r_{j-1} and r_j would freely cancel in the word W. Hence $g_j \neq 1$ since it is a conjugate of h_j in this case. By a similar argument, it follows that if $\varepsilon_{j-1} = -1$ and $\varepsilon_j = 1$, then $g_j \neq 1$. Thus, if $g_j = 1$, then $\varepsilon_{j-1} = \varepsilon_j$ and hence the corresponding t-letters cannot cancel with each other.

Since the normal form for h contains at least one t-letter, Theorem 3.2 gives that $h \notin G$ and hence $L \cap G = H$. Furthermore, by Corollary 3.3,

$$|h|_{A \cup \{t\}} = \sum_{j=1}^{m+1} |g_j|_A + m \geq m. \tag{5}$$

Since each $|h_j|_Z \in \{0, 1\}$, it follows that

$$|h|_Z = \sum_{j=1}^{m+1} |h_j|_Z + m \leq 3m \leq 3|h|_{A \cup \{t\}}.$$

$\square$

By definition, a group is acylindrically hyperbolic if it admits a non-elementary acylindrical action on some hyperbolic metric space. In [49] it is shown that an acylindrically hyperbolical group G always has such an action on a hyperbolic space of the form $\Gamma(G, A)$ where A is a generating set of G.

Definition 3.5. Let A be a generating set of G such that $\Gamma(G, A)$ is hyperbolic and acylindrical. A subgroup $S \leq G$ is called *suitable with respect to A* if the action of S on $\Gamma(G, A)$ is non-elementary and S does not normalize any non-trivial finite subgroup of G.

The following is a modification of [32, Theorem 7.1]; note that condition (a) is a simplification of [32, Theorem 7.1 (d)] and condition (b) is an immediate consequence of the proof of [32, Theorem 7.1]. Recall that $B_A(N)$ denotes the ball of radius N in G with respect to the metric d_A.

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Theorem 3.6. *Let A be a generating set of G such that $\Gamma(G, A)$ is hyperbolic and acylindrical, and let $S \leq G$ be suitable with respect to A. Let $N \in \mathbb{N}$ and $t \in G$. Then there exists a group $\overline{G}$ and a surjective homomorphism $\gamma \colon G \to \overline{G}$ such that*

(a) *There exists a generating set $\overline{A}$ of $\overline{G}$ such that $\gamma(A) \subseteq \overline{A}$ and $\Gamma(\overline{G}, \overline{A})$ is hyperbolic, acylindrical, and non-elementary.*

(b) *$\gamma|_{B_A(N)}$ is injective.*

(c) *$\overline{G} = G/\langle\!\langle t^{-1}w \rangle\!\rangle$ for some $w \in S$. In particular, $\gamma(t) \in \gamma(S)$.*

We are now ready to prove the main result of this subsection.

Proof of Proposition 3.1. Since H is small, there exists a generating set $X \subseteq G$ such that $H \subseteq X$ and $\Gamma(G, X)$ is hyperbolic, acylindrical, and non-elementary. By Theorem 2.6, there exists $g \in G$ loxodromic with respect to the action on $\Gamma(G, X)$. Then $E(g) \hookrightarrow_h (G, X)$ ([32, Corollary 3.16]) and hence $E(g) \hookrightarrow_h (P, X \cup \{t\})$ by Lemma 2.3 and Lemma 2.4.

Let $A = X \cup E(g)$ and let P, L, and Z be as in Lemma 3.4. Obviously Z is bounded with respect to the metric $\mathrm{d}_{X \cup \{t\}}$. By Lemmas 3.4 and 2.12 applied to the group P, hyperbolically embedded subgroup $F = E(g)$, generating set $X \cup \{t\}$, and subgroups $S = G$ and $H = L$, we can chose a generating set Y of P such that $L \subseteq Y$, $\Gamma(P, Y)$ is acylindrical, hyperbolic, and the action of G on $\Gamma(P, Y)$ is non-elementary.

Since $K(G) = \{1\}$, G does not normalize any non-trivial finite normal subgroup of P and hence G is suitable with respect to Y. Since B is finite, we can choose $N \in \mathbb{N}$ such that $B \subseteq B_Y(N)$. Applying Theorem 3.6 gives a group $\overline{P}$ and a surjective homomorphism $\gamma \colon P \to \overline{P}$. By condition (c) of Theorem 3.6 and the obvious Tietze transformation, $\overline{P} \cong G$. In particular, the composition of the natural embedding $G \hookrightarrow P$ and γ is the identity map on G. Let $g = \gamma(t)$ and $K = \gamma(L)$. Since $H \leq L$, $H = \gamma(H) \leq \gamma(L) = K$. By construction $b_i^{-1}ta_i \in L$ for $1 \leq i \leq k$, so the definition of g and K gives that $ga_iK = b_iK$ for $1 \leq i \leq k$. Since $L \subseteq Y$, $K \subseteq \gamma(Y)$ and condition (a) of Theorem 3.6 gives that K is small in G. Since $B \subseteq G$ and B is disjoint from H, B is also disjoint from L by Lemma 3.4. Finally, $L \cup B \subseteq B_Y(N)$, so condition (c) of Theorem 3.6 gives that $B \cap K = \emptyset$. $\qquad\square$

3.3. Proof of Theorem 1.2.

Below we employ the standard notation $x^y = y^{-1}xy$ for elements x, y of a group.

Proof. By Lemma 2.9 it suffices to prove the theorem in the case $K(G) = \{1\}$. Let

$$\bigcup_{k=1}^{\infty} (G^k \times G^k) = \{(\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_2), \ldots\}.$$

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If $\bar{a}_i = (a_1, ..., a_k)$, $\bar{b}_i = (b_1, ..., b_k)$, and $g \in G$, we write $g\bar{a}H = \bar{b}H$ to mean $ga_iH = b_iH$ for all $1 \le i \le k$. We also enumerate all elements of G:

$$G = \{1, g_1, g_2, \ldots\}.$$

We will construct a sequence of subgroups $H_0 \le H_1 \le \ldots$ of G and a sequence of elements $u_1, u_2, \ldots$ of G such that the following conditions hold for every $i \in \mathbb{N}$.

(a) H_i is small in G.

(b) For each $1 \le j \le i$, we have $g_j^{u_j} \notin H_i$.

(c) If $i > 0$ and the triple $(\bar{a}_i, \bar{b}_i, H_i)$ is admissible, then there exists $t_i \in G$ such that $t_i \bar{a}_i H_i = \bar{b}_i H_i$.

We proceed by induction. Set $H_0 = \{1\}$. Assume now that we have already constructed subgroups $H_1 \le \ldots \le H_i$ satisfying (a)–(c) for some $i \ge 0$.

The action of G on the coset space G/H_i is faithful by (a) and Lemma 2.11. Hence there exists $u_{i+1} \in G$ such that $g_{i+1}^{u_{i+1}} \notin H_i$. Let

$$B = \{g_1^{u_1}, \ldots, g_{i+1}^{u_{i+1}}\}.$$

Note that B is disjoint from H_i by part (b) of the inductive assumption and the choice of u_{i+1}.

Let $\bar{a}_{i+1} = (a_1, ..., a_k)$ and $\bar{b}_{i+1} = (b_1, ..., b_k)$. If the triple $(\bar{a}_{i+1}, \bar{b}_{i+1}, H_i)$ is not admissible, we set $H_{i+1} = H_i$. All inductive assumptions obviously hold in this case. Henceforth we assume that $(\bar{a}_{i+1}, \bar{b}_{i+1}, H_i)$ is admissible. Then we can apply Proposition 3.1 to the small subgroup $H = H_i$ and let $H_{i+1} = K$. Conditions (a)–(c) for H_{i+1} follow immediately from the proposition.

Let $H_\infty = \bigcup_{i=1}^\infty H_i$, and let $\Omega = G/H_\infty$ denote the corresponding coset space. We first notice that the action of G on Ω is faithful. Indeed assume that some element g_j acts trivially. Then all conjugates of g_j by elements of G belong to H_∞. In particular, $g_j^{u_j} \in H_\infty$ and hence $g_j^{u_j} \in H_i$ for all sufficiently large i. However this contradicts (b).

Further, for any two collections of pairwise distinct cosets $a_1 H_\infty, ..., a_k H_\infty$ and $b_1 H_\infty, ..., b_k H_\infty$, there exists i such that $\bar{a}_i = (a_1, ..., a_k)$, $\bar{b}_i = (b_1, ..., b_k)$. Since $H_i \le H_\infty$, the triple $(\bar{a}_i, \bar{b}_i, H_i)$ is admissible. Then by (c) we have $t_i a_j H_i = b_j H_i$ for all $1 \le j \le k$, which obviously implies $t_i a_j H_\infty = b_j H_\infty$. Thus the action of G on Ω is highly transitive. $\square$

4 Transitivity degrees of mapping class groups, 3-manifold groups, and graph products

4.1. Preliminaries on permutation groups. We begin by reviewing some basic concepts and facts about multiply transitive groups used in this section. For more details we

refer the reader to [16, 18].

By "countable" we mean finite or countably infinite. Let Ω denote a countable set and let $Sym(\Omega)$ denote the symmetric group on Ω. We will also use the notation $Sym(\mathbb{N})$ for the symmetric group on a countable infinite set. A permutation group $G \leq Sym(\Omega)$ is *k-transitive* if the action of G on Ω is k-transitive; G is *primitive* if it is transitive and does not preserve any non-trivial equivalence relation (or partition) on Ω. Here an equivalence relation on Ω is called *trivial*, if it is equality or if all elements of Ω are equivalent.

In the following two lemmas, we collect some well-known properties of primitive and 2-transitive groups which will be used many times throughout this section. Since the proofs are short and fairly elementary, we provide them for convenience of the reader.

Lemma 4.1. *Let $G \leq Sym(\Omega)$ be a primitive permutation group.*

(a) *Every non-trivial normal subgroup of G is transitive. In particular, if G is infinite then it has no non-trivial finite normal subgroups.*

(b) *For every $\omega \in \Omega$, $Stab_G(\omega)$ is a maximal subgroup of G.*

Proof. Let G be primitive and $N \triangleleft G$. The collection of N-orbits defines a G-invariant partition of Ω, hence by primitivity this partition is trivial. This means that either N acts transitively on Ω or $N = \{1\}$. Further, let $Stab_G(\omega) \leq K \leq G$. Then the G-translates of the orbit $K\omega$ form a G-invariant partition of Ω. By primitivity either $K\omega = \{\omega\}$ or $K\omega = \Omega$. In the former case we have $K = Stab_G(\omega)$; in the latter case $G = K Stab_G(\omega) = K$. $\square$

Recall that a group G has *infinite conjugacy classes*, abbreviated *ICC*, if every nontrivial conjugacy class of G is infinite. Note that ICC groups cannot have non-trivial center or non-trivial finite normal subgroups.

Lemma 4.2. *Let $G \leq Sym(\Omega)$ be a 2-transitive permutation group.*

(a) *G is primitive.*

(b) *If G is infinite, then it is ICC.*

Proof. Part (a) is obvious. To prove (b), let $g \in G \setminus \{1\}$ and let $\omega_0, \omega_1, \ldots$ be distinct elements of Ω. (Note that Ω is infinite since so is G.) Let also σ be an element of Ω such that $g^{-1}(\sigma) \neq \sigma$. Since G is 2-transitive, for all $j \geq 1$ there exists $x_j \in G$ such that $x_j(\omega_0) = g^{-1}\sigma$ and $x_j(\omega_j) = \sigma$. Then $x_j^{-1}gx_j(\omega_0) = \omega_j$, which means that $g^{x_1}, g^{x_2}, \ldots$ are all distinct conjugates of g. Hence G is ICC.

$\square$

The next lemma is also quite elementary. It will be used to bound transitivity degree of various groups from above.

Lemma 4.3. *Let $P \leq Sym(\Omega)$ be an infinite primitive permutation group, T a transitive subgroup of P, and R a non-trivial normal subgroup of P. Suppose that $[T, R] = 1$. Then R is an infinite minimal normal subgroup of P.*

Proof. Let R_0 be a non-trivial subgroup of R such that $R_0 \lhd P$. Consider any $r \in R$ and fix some $\omega \in \Omega$. By part (a) of Lemma 4.1, R_0 is transitive. Hence there exists $r_0 \in R_0$ such that $r_0(\omega) = r(\omega)$. That is, $s(\omega) = \omega$ for $s = r^{-1}r_0$. Since T is transitive, for every $\alpha \in \Omega$ there is $t \in T$ such that $\alpha = t\omega$. Since $[T, R] = 1$ we obtain

$$s(\alpha) = st(\omega) = ts(\omega) = t(\omega) = \alpha.$$

Thus s fixes Ω pointwise, i.e., $s = 1$, which in turn implies $r = r_0 \in R_0$. As this is true for every $r \in R$, we obtain $R_0 = R$. It remains to note that R is infinite by Lemma 4.1 (a). $\square$

Definition 4.4. We say that a group G is *product-like* if G contains non-trivial normal subgroups A, B such that $|A \cap B| = 1$.

The term "product-like" reflects the fact that such groups often occur as subgroups of direct products having non-trivial intersections with multiples.

A group G is said to *virtually* have some property $\mathcal{P}$ if a finite index subgroup of G has $\mathcal{P}$.

Corollary 4.5. *Let G be a countably infinite virtually product-like residually finite group. Then $\mathrm{td}(G) = 1$.*

Proof. Suppose that there exists a faithful, 2-transitive action of G on a set Ω. By our assumption, G contains a finite index subgroup G_0 and non-trivial subgroups $A, B \lhd G_0$ such that $|A \cap B| = 1$. By [18, Theorem 7.2D], a finite index subgroup in an infinite 2-transitive permutation group is primitive. Thus G_0 acts primitively on Ω. By Lemma 4.1 (a), B is transitive. As A and B are normal in G_0, we have $[A, B] \leq A \cap B = \{1\}$. Applying Lemma 4.3 for $P = G_0$, $R = A$, and $B = T$, we obtain that A is an infinite minimal normal subgroup of G_0. However existence of such a subgroup obviously contradicts residual finiteness of G_0. $\square$

Note that there are many 2-transitive solvable groups (e.g., affine groups of fields, see the discussion of affine groups after Question 6.1). However, such groups cannot be residually finite.

Corollary 4.6. *Let G be an infinite residually finite virtually solvable group. Then $\mathrm{td}(G) = 1$.*

Proof. Let G_0 be a solvable normal subgroup of G of finite index and let A be the maximal abelian normal subgroup of G_0. Since G_0 is solvable $A \neq \{1\}$. Since A is characteristic in G_0, it is normal in G. Assume that G acts faithfully and 2-transitively on an infinite set

Ω. Then A is transitive by Lemma 4.1 (a). Since A is abelian, we can apply Lemma 4.3 to $P = G$ and $R = T = A$. As above, we conclude that A is an infinite minimal normal subgroup of G, which contradicts residual finiteness of G. $\qquad\square$

4.2. Mapping class groups Our main reference for mapping class groups is [19]. We review some basic definitions and necessary facts below.

Given $g, n, p \in \mathbb{N} \cup \{0\}$, let $S = \Sigma_{g,n,p}$ be a p times punctured compact, connected, orientable surface of genus g with n boundary components. We also write $\Sigma_{g,p}$ for $\Sigma_{g,0,p}$. By the the *mapping class group* of S, denoted $\mathrm{Mod}(S)$, we mean the group of isotopy classes of orientation preserving homeomorphisms of S; both homeomorphisms and isotopies are required to restrict to the identity on ∂S (however we do allow homeomorphisms to permute the punctures).

Given some $S = \Sigma_{g,n,p}$ with $n \neq 0$, we can glue a once punctured closed disc to each boundary component of S. Let $S' = \Sigma_{g,p+n}$ be the resulting surface. Assume, in addition, that S is not a disc nor an annulus. Then there is a *capping homomorphism* $\kappa \colon \mathrm{Mod}(S) \to \mathrm{Mod}(S')$ whose kernel is generated by Dehn twists corresponding to components of ∂S [19, Sections 3.6.1, 3.6.2]. Recall that a *Dehn twist* T_a around a curve a commutes with every $f \in \mathrm{Mod}(S)$ fixing a. In particular, this implies that $Ker(\kappa)$ is a central subgroup of $\mathrm{Mod}(S)$.

We say that a surface $S = \Sigma_{g,p}$ is *exceptional* if $3g + p \leq 4$. Thus exceptional surfaces are spheres with at most 4 punctures and tori with at most 1 puncture. The following is a complete list of mapping class groups of exceptional surfaces (see [19, Chapter 2]):

$$\mathrm{Mod}(\Sigma_{1,0}) = \mathrm{Mod}(\Sigma_{1,1}) = SL(2, \mathbb{Z}),$$

$$\mathrm{Mod}(S_{0,0}) = \mathrm{Mod}(S_{0,1}) = \{1\},$$

and

$$\mathrm{Mod}(\Sigma_{0,2}) = \mathbb{Z}_2, \quad \mathrm{Mod}(\Sigma_{0,3}) = S_3, \quad \mathrm{Mod}(\Sigma_{0,4}) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes PGL(2, \mathbb{Z}).$$

Mapping class groups of non-exceptional surfaces are acylindrically hyperbolic since they act acylindrically on the corresponding curve complexes, which are hyperbolic [37, 11]; mapping class groups of exceptional surfaces are either finite or non-virtually cyclic and hyperbolic. Hence we obtain the following (see Section 8 of [49]).

Lemma 4.7. $\mathrm{Mod}(\Sigma_{g,p})$ *is acylindrically hyperbolic for all* $(g, p) \notin \{(0,0), (0,1), (0,2), (0,3)\}$.

Thus to describe highly transitive mapping class groups we need to know which mapping class groups contain non-trivial finite normal subgroups. The following lemma is probably known to specialists. However we were unable to find a reference, so we prove it here. The idea of the proof was suggested by Dan Margalit.

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Lemma 4.8. *If $3g + p \geq 5$ and $(g, p) \neq (2, 0)$, then $\mathrm{Mod}(\Sigma_{g,p})$ has no non-trivial finite normal subgroups.*

Proof. For basic properties of Dehn twists used in this proof we refer to [19, Chapter 3]. Recall also that under our assumptions the action of $\mathrm{Mod}(\Sigma_{g,p})$ on the curve complex is faithful [9, 54].

Assume that an element $f \in \mathrm{Mod}(\Sigma_{g,p})$ belongs to a finite normal subgroup N and let c be an essential simple closed curve on $\Sigma_{g,p}$. By the classification of subgroups generated by two distinct Dehn twists (see [19, p. 86]), the subgroup $H = \langle T_c^2, T_{f(c)}^2 \rangle$ is isomorphic to one of the groups $\mathbb{Z}$, $\mathbb{Z}^2$, or the free group of rank 2. Note that $H \cap N = \{1\}$ since neither of the groups from the above list has torsion. On the other hand, $[T_c^2, f] = T_c^2 f T_c^{-2} f^{-1} = T_c^2 T_{f(c)}^{-2} \in H \cap N$ since N is normal. Therefore $[T_c^2, f] = 1$, which implies that $f(c) = c$ (see [19, Section 3.3]). Since this is true for every essential simple closed curve c, f acts trivially on the curve complex of $\Sigma_{g,p}$ and consequently $f = 1$. $\qquad \square$

Corollary 4.9. $\mathrm{Mod}(\Sigma_{g,n,p})$ *admits a highly transitive faithful action on a countably infinite set if and only if $n = 0$, $3g + p \geq 5$, and $(g, p) \neq (2, 0)$.*

Proof. Let $G = \mathrm{Mod}(\Sigma_{g,n,p})$. If $n \neq 0$, then either G is trivial (if $\Sigma_{g,n,p}$ is a closed disk or a closed annulus) or has a non-trivial central subgroup (the kernel of the capping homomorphism) and hence $\mathrm{td}(G) = 1$. Similarly $\mathrm{td}(G) = 1$ if $(g, p) = (2, 0)$ since $Z(G) = \mathbb{Z}_2$ in this case. Further if $n = 0$ and $\Sigma_{g,n,p}$ is exceptional, then G is either trivial or has a non-trivial finite normal subgroup, so it cannot have a highly transitive faithful action on a countably infinite set again. In all remaining cases, G is acylindrically hyperbolic and has trivial finite radical by the previous lemma, so the result follows from Theorem 1.2. $\qquad \square$

Let us now discuss subgroups of mapping class groups. The following observation will be used several times.

Remark 4.10. Let $S = \Sigma_{g,n,p}$. By gluing $\Sigma_{1,1,0}$ to every boundary component of S, we obtain an embedding $\mathrm{Mod}(S) \to \mathrm{Mod}(\Sigma_{g+n,p})$. Thus, if we want if we want to prove certain claim for subgroups of $\mathrm{Mod}(\Sigma_{g,n,p})$, it suffices to prove it for $n = 0$.

Lemma 4.11. *Let $G \leq \mathrm{Mod}(\Sigma_{g,n,p})$. Assume that G is non-trivial and ICC. Then G is either virtually product-like or acylindrically hyperbolic with trivial finite radical.*

Proof. By Remark 4.10, we can assume that $G \leq \mathrm{Mod}(\Sigma_{g,p})$. We consider two cases.

Case 1. If G contains a pseudo-Anosov element a, then it is either virtually cyclic or acylindrically hyperbolic. Indeed in the non-exceptional case it follows from the fact that a is contained in a virtually cyclic hyperbolically embedded subgroup of G [15, Theorem 2.19] and Theorem 2.8. Since G is ICC, it cannot be virtually cyclic and cannot contain non-trivial finite normal subgroups. Thus G is acylindrically hyperbolic with trivial finite radical.

Case 2. Now assume that G does not contain a pseudo-Anosov element. Then by Ivanov's theorem [34] G is reducible, i.e., it fixes a multicurve. Moreover, in this case there is a multicurve c and a finite index subgroup $G_0 \leq G$ with the following properties. Let $S_1, \ldots, S_k$ be the connected components of $S - c$. We naturally think of S_i as a punctured closed surface (cuts along components of c give rise to new punctures). Then G_0 fixes every S_i setwise and the restriction of G_0 to S_i is either trivial or infinite irreducible. This gives a homomorphism

$$\rho \colon G_0 \to \mathrm{Mod}(S_1) \times \cdots \times \mathrm{Mod}(S_k), \tag{6}$$

whose kernel is generated by Dehn twists around components of c [19, Proposition 3.20]. In particular, $Ker(\rho)$ is central in G_0. Since G is ICC, so is G_0 and hence ρ is injective.

Let $G_1 = \rho(G_0) \cap \mathrm{Mod}(S_1)$ and $G_2 = \rho(G_0) \cap (\mathrm{Mod}(S_2) \times \cdots \times \mathrm{Mod}(S_k))$. If both G_1 and G_2 are non-trivial, then G is product-like. If $G_2 = \{1\}$, then the composition of ρ and the projection map to the first factor in (6) is injective and its image is either trivial or irreducible; in the latter case we complete the proof as in the first case. If $G_1 = \{1\}$, then the composition of ρ with the projection to $\mathrm{Mod}(S_2) \times \cdots \times \mathrm{Mod}(S_k)$ is injective and we complete the proof by induction. $\square$

We will also need the following result. It is well-known to specialists (see, e.g., the remark in the end of Section 6.4.3 of [19]), but we were unable to locate the proof in the literature. We are grateful to Dan Margalit for explaining the argument to us.

Lemma 4.12. *The group* $\mathrm{Mod}(\Sigma_{g,n,p})$ *is residually finite for all* $g, n, p \in \mathbb{N} \cup \{0\}$.

Proof. Since a subgroup of a residually finite group is residually finite, we can assume that $n = 0$ by Remark 4.10. For $p = 0$, residual finiteness of $\mathrm{Mod}(\Sigma_{g,p})$ is a classical result of Grossman [25]. Now assume that $p > 0$. It is easy to see that any (residually finite)-by-finite group is residually finite. Thus it suffices to prove residual finiteness of $\mathrm{PMod}(\Sigma_{g,p})$, the subgroup of all elements of $\mathrm{Mod}(\Sigma_{g,p})$ that fix the punctures. The latter group has a faithful representation by automorphisms of $\pi_1(\Sigma_{g,p-1})$ [8]. Since $\pi_1(\Sigma_{g,p-1})$ is residually finite and the automorphism group of a residually finite finitely generated group is residually finite [5], we obtain that $\mathrm{PMod}(\Sigma_{g,p})$ is residually finite, and hence so is $\mathrm{Mod}(\Sigma_{g,p})$. $\square$

The next corollary refers to conditions (C_1)-(C_3) stated in the introduction.

Corollary 4.13. *Let* G *be an infinite subgroup of* $\mathrm{Mod}(\Sigma_{g,n,p})$. *Then* $\mathrm{td}(G) \in \{1, \infty\}$ *and conditions* (C_1)-(C_3) *are equivalent. Moreover,* (C_1)-(C_3) *are also equivalent to*

(C_4) G *is non-trivial, ICC, and not virtually product-like.*

Proof. We first note that $(C_4) \Rightarrow (C_3)$ by Lemma 4.11. Further, $(C_3) \Rightarrow (C_2)$ by Theorem 1.2 and the implication $(C_2) \Rightarrow (C_1)$ is obvious. Assume now that G does not satisfy (C_4). If G is not ICC, then $\mathrm{td}(G) = 1$ by part (b) of Lemma 4.2. If G is virtually product-like, then $\mathrm{td}(G) = 1$ by Lemma 4.12 and Corollary 4.5. Thus $\mathrm{td}(G) = 1$ whenever G does

not satisfy (C_4). In particular, we obtain $(C_1) \Rightarrow (C_4)$. This completes the proof of the equivalence of (C_1)-(C_3) and also shows that $\mathrm{td}(G) \in \{1, \infty\}$. $\square$

4.3. 3-manifold groups. Let G be a subgroup of the fundamental group of a compact 3-manifold. It was proved in [41, Theorem 5.6] that one of the following three (mutually exclusive) conditions holds:

(I) G is acylindrically hyperbolic with trivial finite radical.

(II) G contains an infinite cyclic normal subgroup Z and G/Z is acylindrically hyperbolic.

(III) G is virtually polycyclic.

Moreover, if G is itself a fundamental group of a compact irreducible 3-manifold M (this is always the case if G is finitely generated by the Scott's core theorem), then we can replace condition (II) with a more geometric one.

(II*) M is Seifert fibered.

Corollary 4.14. *Let G be an infinite subgroup of the fundamental group of a compact 3-manifold. Then $\mathrm{td}(G) \in \{1, \infty\}$ and conditions (C_1)–(C_3) are equivalent.*

Proof. This follows immediately from Theorem 1.2 and the above trichotomy. Indeed if G satisfies (I), then $\mathrm{td}(G) = \infty$ by Theorem 1.2. If G satisfies (II), it is not ICC (note that the conjugacy class of any element $z \in Z$ is contained in $\{z, z^{-1}\}$) and hence $\mathrm{td}(G) = 1$ by part (b) of Lemma 4.2. Finally if G satisfies (III), then G cannot act primitively on a countable set by Lemma 4.6 and hence $\mathrm{td}(G) = 1$. Thus conditions (C_1)–(C_3) hold simultaneously in case (I) and fail simultaneously in cases (II) and (III). $\square$

Similarly, if G is itself a fundamental group of a compact irreducible 3-manifold M, we obtain the following.

Corollary 4.15. *Let M be a compact irreducible 3-manifold. Then $\pi_1(M)$ admits a highly transitive faithful action on a countably infinite set iff it is not virtually solvable and M is not Seifert fibered.*

4.4. Graph products. It is proved in [41] that for every subgroup G of a finite graph product $\Gamma\{G_v\}_{v \in V}$, one of the following conditions holds.

(a) There is a short exact sequence $\{1\} \to K \to G \to S \to \{1\}$, where K is finite and S is isomorphic to a subgroup of some G_v.

(b) G is virtually cyclic;

(c) G contains two infinite normal subgroups $N_1, N_2 \lhd G$ such that $|N_1 \cap N_2| < \infty$.

(d) G is acylindrically hyperbolic.

We will also need a result of Cameron.

Lemma 4.16 ([16, Corollary 2.2]). *Suppose that $G \leq Sym(\mathbb{N})$ is infinite and k-transitive for some $k \geq 2$. Let N be a non-trivial normal subgroup of G. Then either N is $(k-1)$-transitive or $k = 3$ and N is an elementary abelian 2-group (i.e., a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$).*

Similar to the previous cases, we obtain the following.

Corollary 4.17. *Let $G \leq \Gamma\{G_v\}_{v \in V}$ be a countably infinite subgroup of a graph product of a finite family of groups G_v, $v \in V$. Suppose that G is not isomorphic to a subgroup of some G_v. Then $\mathrm{td}(G) \in \{1, 2, \infty\}$ and conditions (C_1)–(C_3) are equivalent. If, in addition, every G_v is residually finite, then $\mathrm{td}(G) \in \{1, \infty\}$.*

Proof. If G satisfies (a), then K must be non-trivial and hence $\mathrm{td}(G) = 1$ by part (b) of Lemma 4.2. Similarly $\mathrm{td}(G) = 1$ in the case (b) as non-trivial virtually cyclic groups are not ICC. Suppose (c) holds. If $N_1 \cap N_2 \neq \{1\}$, then G contains a non-trivial finite normal subgroup and we obtain $\mathrm{td}(G) = \{1\}$ as above. Thus it remains to consider the case $N_1 \cap N_2 = \{1\}$. We are going to prove that $\mathrm{td}(G) \leq 2$ in this case.

Arguing by contradiction, suppose that $\mathrm{td}(G) \geq 3$. Let N denote the (normal) subgroup of G generated by N_1 and N_2. Clearly $N \cong N_1 \times N_2$. Let G act faithfully and 3-transitively on a set Ω. According to Lemma 4.16, there are two possibilities.

Case 1. Neither of the subgroups N_1, N_2 is 2-transitive. Then N is an elementary abelian 2-group, so we can apply Lemma 4.3 to $T = R = N$ and conclude that N is a minimal normal subgroup of G. However N_i, $i = 1, 2$ are proper subgroups of N. Contradiction.

Case 2. Assume now that one of the subgroups N_1, N_2, say N_2, is 2-transitive. Let $\alpha, \beta, \gamma \in \Omega$ be distinct elements. N_1 is transitive by Lemma 4.1. Hence there exists $a \in N_1$ such that $a(\alpha) = \beta$. Since N_2 is 2-transitive, there exists $b \in N_2$ such that $b(\alpha) = \alpha$ and $b(\beta) = \gamma$. Then $ab(\alpha) = \beta$ while $ba(\alpha) = \gamma$. This contradicts $[N_1, N_2] \leq N_1 \cap N_2 = \{1\}$.

Notice that by a theorem of Green [23] (see also [31]), a finite graph product of residually finite groups is residually finite. Thus if all multiples are residually finite, so is G and instead of considering cases 1,2 above we can immediately conclude that $\mathrm{td}(G) = 1$ by Corollary 4.5.

Finally, if G satisfies (d), the conclusion follows from Theorem 1.2 and part (b) of Lemma 4.2. $\square$

Note that the case $\mathrm{td}(G) = 2$ in the above corollary does realize. The following example is well-known.

Example 4.18. Let C be a countably infinite group where all non-trivial elements are conjugate. First examples of countable groups of this kind were constructed by G. Higman, B. Neumann and H.Neumann (see [36]); finitely generated examples can be found in [51]. Let $G = C \times C$. Then G naturally acts on C by the rule

$$(a, b)c = acb^{-1}.$$

Note that this action is isomorphic to the action of G on cosets G/D, where $D = \{(c, c) \mid c \in C\}$. It is easy to see that this action is 2-transitive. Indeed let $(x_1, y_1), (x_2, y_2)$ be two pairs of distinct elements of C. Since all nontrivial elements of C are conjugate, there exists $b \in C$ such that $x_2^{-1} y_2 = b^{-1} x_1^{-1} y_1 b$. Let also $a = x_2 b^{-1} x_1^{-1}$. Obviously we have $ax_1 b = x_2$ and

$$ay_1 b = x_2 b^{-1} x_1^{-1} y_1 b = x_2 x_2^{-1} y_2 = y_2.$$

Finally we observe that if $acb^{-1} = c$ for every $c \in C$, then substituting $c = 1$ we obtain $a = b$. Hence $c^{-1} ac = a$ for all $c \in C$, i.e., $\{a\}$ is a conjugacy class. Clearly this implies $a = 1$ and consequently $b = 1$. Thus the action is faithful and we have $\mathrm{td}(G) \geq 2$. On the other hand, if $\mathrm{td}(G) \geq 3$, then both multiples of $G = C \times C$ act 2-transitively by Lemma 4.16 and we get a contradiction arguing as as in Case 2 of the proof of Corollary 4.17. Thus $\mathrm{td}(G) = 2$.

If G is a finite graph product of copies of $\mathbb{Z}$, i.e., a right angled Artin group, then it is either $\mathbb{Z}$, of decomposes as a direct product of two non-trivial groups, or is acylindrically hyperbolic [41]. Since G is torsion free, its finite radical is trivial and we obtain the following.

Corollary 4.19. *A right angled Artin group is highly transitive if and only if it is non-cyclic and directly indecomposable.*

4.5. Miscellaneous results. Here we discuss applications of our main theorem to some other classes of acylindrically hyperbolic groups.

Lemma 4.20. *Let G be a group hyperbolic relative to a collection of proper subgroups. Then G is either acylindrically hyperbolic or virtually cyclic.*

Proof. Let G be hyperbolic with respect to a collection of proper subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. We accept the definition of a relatively hyperbolic group from [50]. In particular, we do not assume that G is countable. Withoul loss of generality we can assume that all H_λ are non-trivial.

We consider two cases.

Case 1. There exists an infinite $H \in \{H_\lambda\}_{\lambda \in \Lambda}$. By [15, Proposition 4.28] we have $H \hookrightarrow_h G$. Since H is infinite and $H \neq G$, it is non-degenerate and hence G is acylindrically hyperbolic by Theorem 2.8.

Case 2. Suppose now that all H_λ are finite. Again there are two possibilities. First assume that Λ is finite. Then by [50, Theorem 2.40] G is an (ordinary) hyperbolic group.

Hence G is either virtually cyclic or acylindrically hyperbolic. Finally assume that Λ is infinite. Then by part 1) of [50, Theorem 2.44] G decomposes as a non-trivial free product of infinitely many non-trivial groups. It is easy to see that the action on the associated Bass-Serre tree is acylindrical and non-elementary. Thus G is acylindrically hyperbolic in this case. $\qquad\square$

From this lemma, Theorem 1.2 and Lemma 4.2(b) we obtain the following.

Corollary 4.21. *Let G be a countable group hyperbolic relative to a collection of proper subgroups. Then G is highly transitive if and only if it is not virtually cyclic and has no non-trivial finite normal subgroups.*

Corollary 4.22. *$Out(F_n)$ is highly transitive if and only if $n \geq 3$.*

Proof. Recall that $Out(F_n)$ is acylindrically hyperbolic for $n \geq 2$ [15, 49]. $Out(F_2) = GL_2(\mathbb{Z})$ has non-trivial finite radical, hence $\mathrm{td}(Out(F_2)) = 1$ by Lemma 4.2 (b). On the other hand, for $n \geq 3$, $Out(F_n)$ has trivial finite (and even amenable) radical. This follows immediately, for example, from C^*-simplicity of $Out(F_n)$ for $n \geq 3$ proved in [10]. This fact can also be derived from results of [56]. $\qquad\square$

Corollary 4.23. *Every 1-relator group with at least 3 generators is highly transitive.*

Proof. Let G be a 1-relator group G with at least 3 generators. It was proved in [41] that G is acylindrically hyperbolic. Thus we only need to show that G has trivial finite radical. It is shown in [52] (see the proof of Theorem III there) that every 1-relator group with at least 3 generators splits as an HNN-extension $G = H_{A^t=B}$, where A, B are proper subgroups of H, and there exists $h \in H$ such that $A \cap B^h = \{1\}$. Let T be the Bass-Serre tree associated to this decomposition and let $N \lhd G$ be a finite normal subgroup. Recall that every finite group acting on a tree without inversions fixes a vertex. Hence N fixes a vertex of T. But since N is normal and the action of G is transitive, N fixes all vertices of T. In particular, $N \leq A \cap B^h = \{1\}$. $\qquad\square$

5 Highly transitive actions and mixed identities

5.1. Mixed identity free groups. We begin by introducing notation, which will be used throughout the rest of the paper. Let F_n denote the free group of rank n with basis $x_1, \ldots, x_n$. Given a group G, we identify elements of the free product $G * F_n$ with their normal forms (see Theorem 3.2). Thus we think of elements of $G * F_n$ as words in the alphabet $\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\} \cup G$. Given $w \in G * F_n$ and any elements $a_1, \ldots, a_n \in G * F_n$, we denote by $w(a_1, \ldots, a_n)$ the word obtained from w by substituting a_i for x_i for all $i = 1, \ldots, n$. Alternatively, we can describe $w(a_1, \ldots, a_n)$ as the image of w under the homomorphism $G * F_n \to G * F_n$ that is identical on G and sends x_i to a_i, $i = 1, \ldots, n$.

Recall that a group G is said to satisfy a *mixed identity* $w = 1$ for some $w \in G * F_n$ if $w(g_1, \ldots, g_n) = 1$ for all $g_1, \ldots, g_n \in G$. We say that a mixed identity $w = 1$ is *non-trivial* if $w \neq 1$ as an element of $G * F_n$.

Remark 5.1. It is easy to see that if $G \neq \{1\}$, then for every $n \in \mathbb{N}$ there is an embedding $G * F_n \to G * \langle x \rangle$ which is identical on G. (E.g., using normal forms in free products it is easy to show that the subgroup of $G * \langle x \rangle$ generated by G, xgx, $\ldots$, $x^n gx^n$ is naturally isomorphic to $G * F_n$ for every $g \in G \setminus \{1\}$.) Thus G satisfies some non-trivial mixed identity $w \in G * F_n$ if and only if it satisfies a non-trivial mixed identity $w' \in G * \langle x \rangle$ (i.e., with one variable).

Definition 5.2. A group G is called *mixed identity free* (or MIF, for brevity) if it does not satisfy any non-trivial mixed identity.

Recall that, for a given group G, a *G-group* is a group H with a fixed embedding $G \leq H$. Let $\mathcal{L}_G$ denote the first order language of G-groups. Thus the alphabet of $\mathcal{L}_G$ consists of the standard logical symbols, the group operations (multiplication and taking inverse), and constants corresponding to elements of G. Let $Th_\forall^G(H)$ denote the universal theory of H as a G-group. That is, $Th_\forall^G(H)$ is the set of all universal sentences in $\mathcal{L}_G$ that hold true in H. Two G-groups H_1, H_2 are *universally equivalent* if $Th_\forall^G(H_1) = Th_\forall^G(H_2)$. In general, being universally equivalent as G-groups is stronger that being universally equivalent in the usual sense (i.e., as $\{1\}$-groups). For details on G-groups and their elementary theory we refer to [7].

Proposition 5.3. *A countable group G is MIF if and only if G and $G * F_n$ are universally equivalent as G-groups for all $n \in \mathbb{N}$.*

Proof. The 'if' part is obvious since satisfying a non-trivial mixed identity can be encoded by a universal sentence in $\mathcal{L}_G$. Thus we only need to prove the 'only if' part. Assume that G is MIF. Fix any $n \in \mathbb{N}$. Let $G * F_n = \{1, w_1, w_2, \ldots\}$. Let $G * F_n \to G * F_{n+1} = G * F_n * \langle x \rangle$ be the natural embedding. Let $u_1 = w_1$ and for $i \geq 2, \ldots$, let

$$u_i = [w_1, x^{-1} w_2 x, \ldots, x^{-(i-1)} w_i x^{i-1}],$$

where the commutator is defined inductively by

$$[a_1, a_2] = a_1^{-1} a_2^{-1} a_1 a_2 \quad \text{and} \quad [a_1, \ldots, a_k, a_{k+1}] = [[a_1, \ldots, a_k], a_{k+1}].$$

Using normal forms, it is easy to see that u_i is a non-trivial element of $G * F_{n+1}$. Since G is MIF, for every i, there exist $g_{i1}, \ldots, g_{i,n+1} \in G$ such that $u_i(g_{i1}, \ldots, g_{i,n+1}) \neq 1$. In particular, this implies that $w_m(g_{i1}, \ldots, g_{in}) \neq 1$ for all $m \leq i$.

Let ω be a non-principal ultrafilter on $\mathbb{N}$ and let G^ω denote the corresponding ultrapower of G. Let g_j denote the element of G^ω represented by the sequence $(g_{ij})_{i \in \mathbb{N}}$, $j = 1, \ldots, n$. We think of G^ω as a G-group, where the fixed embedding of G is the diagonal one. Let $\varepsilon \colon G * F_n \to G^\omega$ be the homomorphism whose restriction on G is the diagonal embedding

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and such that x_j is mapped to g_j for $j = 1, \ldots, n$. By the choice of elements g_{ij}, we obtain that for every m, $w_m(g_{i1}, \ldots, g_{in}) \neq 1$ for all but finitely many i. Since ω is non-principal, this implies $\varepsilon(w_m) \neq 1$ for every m. Hence ε is injective. Thus we obtain a sequence of embeddings of G-groups: $G \to G * F_n \to G^\omega$. By the Los theorem, G and G^ω have the same elementary (in particular, universal) theory as G-groups. Since universal properties of algebraic structures are inherited by substructures, we obtain

$$Th_\forall^G(G) = Th_\forall^G(G^\omega) \subseteq Th_\forall^G(G * F_n) \subseteq Th_\forall^G(G).$$

Consequently, $Th_\forall^G(G * F_n) = Th_\forall^G(G)$. $\qquad\qquad\square$

The next proposition shows that being MIF is a much stronger property than not satisfying any nontrivial identity. Recall that the *girth* of a finitely generated group G is the supremum of all $n \in \mathbb{N}$ such that for every finite generating set X of G, the Cayley graph $\Gamma(G, X)$ contains a cycle of length at most n without self-intersections [53].

Proposition 5.4. *For any MIF group, the following conditions hold.*

(a) *G is ICC. In particular, G has trivial center and contains no non-trivial finite normal subgroups.*

(b) *G is not decomposable as direct products of two non-trivial groups.*

(c) *Every non-trivial subnormal subgroup of G is MIF.*

(d) *If G is finitely generated, then G has infinite girth.*

Proof. (a) Let $a \neq 1$ be an element whose conjugacy class in G consists of n elements. Then G satisfies the mixed identity $[x^{n!}, a] = 1$.

(b) Let $G = A \times B$. Fix any elements $a \in A$, $b \in B$. Then G satisfies the mixed identity $[[a, x], [b, x]] = 1$, which is obviously non-trivial provided $a \neq 1$ and $b \neq 1$.

(c) Clearly it suffices to prove that every nontrivial normal subgroup $N \lhd G$ is MIF. Let $a \in N \setminus \{1\}$. Assume that N satisfies an identity $w(x) = 1$ for some $w(x) \in N * \langle x \rangle$. Since N is normal, G satisfies the identity $w([a, x]) = 1$. Since G is MIF, $w([a, x]) = 1$ as an element of $G * \langle x \rangle$. By using normal forms, it is easy to see that $N * \langle x \rangle$ is isomorphic to the subgroup of $N * \langle x \rangle$ generated by N and $[a, x]$ via the map that takes x to $[a, x]$ and is identical on N. This map takes $w([a, x])$ to $w(x)$ and hence $w(x) = 1$ as an element of $N * \langle x \rangle$. Thus N can only satisfy the trivial identity.

(d) Suppose that G has finite girth k. Then for every finite generating set $g_1, \ldots, g_n$ of G, there exists a non-trivial element $w \in F_n$ of length at most k such that $w(g_1, \ldots, g_n) = 1$ in G. Consider any generating set $a_1, \ldots, a_n$ of $G * \langle x \rangle$ and let $a_i(g)$ denote the image of a_i under the epimorphism of $G * \langle x \rangle$ that is identical on G and sends x to an element

$g \in G * \langle x \rangle$. Since $a_1(g), \ldots, a_n(g)$ generate G for every $g \in G$, the group G satisfies the universal sentence

$$\forall g \; w_1(a_1(g), \ldots, a_n(g)) = 1 \vee \ldots \vee w_s(a_1(g), \ldots, a_n(g)) = 1,$$

where $w_1, \ldots, w_s$ is the list of all non-trivial elements of F_n of length at most k. By Proposition 5.3 we obtain that the same sentence holds true in $G * \langle x \rangle$. In particular, it is true for $g = x$. Since $a_i(x) = a_i$, we obtain that there is a non-trivial relation of length at most k between $a_1, \ldots, a_n$. This is true for any generating set $a_1, \ldots, a_n$ of $G * \langle x \rangle$, so we obtain that $G * \langle x \rangle$ has finite girth. However this is well-known to be false (for example, see [6, Example 2.15]). □

Remark 5.5. The proof of Proposition 5.4(d) actually shows that for all k, G has a generating set with n elements and no simple cycles of length $\leq k$, where n is independent of k. This is equivalent to G *preforming* the free group F_n in the language of [6] and to G being without *almost-identities* in the language of [47]. However, it is unclear if this is actually stronger then the property of infinite girth (see [6, Question 8.5]).

5.2. A dichotomy for highly transitive groups

Lemma 5.6. *Let $s \in Sym(\mathbb{N})$ be an arbitrary permutation, X a finite subset of $\mathbb{N}$. Then for any permutations $a_1, \ldots, a_k \in Sym(\mathbb{N})$ with infinite supports and any $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$, there exists $t \in Sym(\mathbb{N})$ such that $t|_X \equiv s|_X$ and $a_k t^{\alpha_k} \cdots a_1 t^{\alpha_1}(n) \neq n$ for some $n \in \mathbb{N}$.*

Proof. The idea of the proof is to construct a sequence of pairwise distinct numbers $n_0, m_1, n_1, \ldots, m_k, n_k \in \mathbb{N}$ and a permutation $t \in Sym(\mathbb{N})$ such that $t|_X \equiv s|_X$ and the elements t^{α_i} and a_i act as follows:

$$n_0 \xrightarrow{t^{\alpha_1}} m_1 \xrightarrow{a_1} n_1 \xrightarrow{t^{\alpha_2}} m_2 \xrightarrow{a_2} n_2 \xrightarrow{t^{\alpha_3}} \ldots n_{k-1} \xrightarrow{t^{\alpha_k}} m_k \xrightarrow{a_k} n_k \tag{7}$$

Then clearly $a_k t^{\alpha_k} \cdots a_1 t^{\alpha_1}(n_0) = n_k \neq n_0$.

We will first construct the desired sequence $n_0, m_1, n_1, \ldots, m_k, n_k$ by induction. Let n_0 be any element of $\mathbb{N} \setminus (X \cup s(X))$. Further for $k \geq i \geq 1$, we let

$$Y_i = X \cup s(X) \cup \{n_0\} \cup \{m_j, n_j \mid 1 \leq j \leq i - 1\}$$

and chose any

$$m_i \in supp(a_i) \setminus (Y_i \cup a_i^{-1}(Y_i)). \tag{8}$$

Such a choice is always possible since Y_i is finite while $supp(a_i)$ is infinite by our assumption. We let $n_i = a_i(m_i)$. Note that (8) guarantees that $n_i \neq m_i$ and $\{n_i, m_i\}$ is disjoint from Y_i. This completes the inductive step. Clearly the resulting sequence $n_0, m_1, n_1, \ldots, m_k, n_k$ consists of pairwise distinct numbers and does not contain any elements from $X \cup s(X)$.

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Further, for each $i = 1, \ldots, k$, we fix arbitrary natural numbers l_{ij}, $1 \leq j \leq |\alpha_i| - 1$, such that all l_{ij} are pairwise distinct and are not contained in Y_k. Again this is possible since Y_k is finite. For $i = 1, \ldots, k-1$, we define a cycle

$$c_i = (n_{i-1}, l_{i1}, \ldots, l_{i,|\alpha_i|-1}, m_i)^{sgn(\alpha_i)}$$

(if $|\alpha_i| = 1$, then $c_i = (n_{i-1}, m_i)^{sgn(\alpha_i)}$). We also denote by s_0 any permutation from $Sym(\mathbb{N})$ with support $supp(s_0) \subseteq X \cup s(X)$ such that $s_0(x) = s(x)$ for all $x \in X$. We now define

$$t = s_0 c_1 \cdots c_k.$$

Note that the sets $supp(c_1), \ldots, supp(c_k)$, and X are pairwise disjoint by the choice of elements m_i, n_i, and l_{ij}. Hence we have $t(x) = s_0(x) = s(x)$ for every $x \in X$, and $t^{\alpha_i}(n_{i-1}) = c_i^{\alpha_i}(n_{i-1}) = m_i$ for $i = 1, \ldots, k$. Thus t^{α_i} and a_i act as prescribed by the diagram (7). □

Recall that a subset of a topological space is *comeagre* if it is the intersection of countably many sets with dense interiors. Since $Sym(\mathbb{N})$ is a Polish space with respect to the topology of pointwise convergence, every comeagre subset of $Sym(\mathbb{N})$ is dense (and, in particular, non-empty) by the Baire Category Theorem. It is customary to think of comeagre subsets as subsets including "most" elements of the ambient space.

Definition 5.7. We say that a certain property *holds for a generic element* of a topological space T if it holds for all elements of a comeagre subset of T.

Corollary 5.8. *Let G be a highly transitive subgroup of $Sym(\mathbb{N})$. Then either G contains $Alt(\mathbb{N})$ or for a generic element $t \in Sym(\mathbb{N})$, the subgroup generated by G and t is naturally isomorphic to $G * \langle t \rangle$.*

Proof. Recall that by the Wielandt theorem [29, Theorem 5.2] every infinite primitive subgroup of $Sym(\mathbb{N})$ containing a non-trivial permutation with finite support contains $Alt(\mathbb{N})$. Thus we can assume that every non-trivial element of G has infinite support.

Let w be a non-trivial element of the free product $G * \langle x \rangle$. Recall that for $g \in G$, $w(g)$ denotes the image of w under the homomorphism $G * \langle x \rangle \to G$ that is identical on G and sends x to g. Note that the set of solutions of any equation in a topological group is closed and hence the set $A_w = \{g \in Sym(\mathbb{N}) \mid w(g) \neq 1\}$ is open. By Lemma 5.6, A_w is also dense; indeed the lemma claims that we can find an element from A_w in any open neighborhood of any element $s \in G$. Hence the set

$$A = \bigcap_{w \in G * \langle x \rangle \setminus \{1\}} A_w$$

is comeager. Clearly $\langle G, t \rangle \cong G * \langle t \rangle$ for all $t \in A$. □

We are now ready to prove the main result of this section.

Theorem 5.9. *Let G be a highly transitive countable group. Then one of the following two mutually exclusive conditions holds.*

(a) G contains a normal subgroup isomorphic to $Alt(\mathbb{N})$.

(b) G is MIF.

Proof. We identify G with a dense subgroup of $Sym(\mathbb{N})$. By Corollary 5.8, either G contains a normal subgroup isomorphic to $Alt(\mathbb{N})$ or there exists $t \in Sym(\mathbb{N})$ such that the subgroup generated by G and t is naturally isomorphic to $G * \langle t \rangle$. In the latter case G cannot satisfy any non-trivial mixed identity. Indeed, by Remark 5.1 it suffices to show that G does not satisfy any non-trivial mixed identity with one variable. Let $w \in G * \langle x \rangle \setminus \{1\}$ and suppose that $w(g) = 1$ for all $g \in G$. Then $w(g) = 1$ for all $g \in Sym(\mathbb{N})$ since the set of solutions of any equation in a topological group is closed and G is dense in $Sym(\mathbb{N})$. However $w(t) \neq 1$, a contradiction.

It remains to prove that conditions (a) and (b) are mutually exclusive. Assume that G contains $Alt(\mathbb{N})$ as a normal subgroup. Let $a = (1,2,3)$, $b = (4,5,6)$, $c = (7,8,9)$, $d = (10,11,12)$. Then for every $g \in Alt(\mathbb{N})$, the elements $g^{-1}ag$, $g^{-1}bg$, $g^{-1}cg$, and $g^{-1}dg$ are 3-cycles with disjoint supports. Hence the support of at least one of them is disjoint from $\{1,2,3\}$. This means that at least one of the commutators $[a^g, a]$, $[b^g, a]$, $[c^g, a]$, $[d^g, a]$ is trivial. Thus $Alt(\mathbb{N})$ satisfies the universal sentence

$$\forall g \; ([a^g, a] = 1 \vee [b^g, a] = 1 \vee [c^g, a] = 1 \vee [d^g, a] = 1), \tag{9}$$

which obviously fails in $Alt(\mathbb{N}) * \langle x \rangle$ for $g = x$. By Proposition 5.3, this implies that $Alt(\mathbb{N})$ is not MIF. (In this particular case we can easily prove this fact directly by taking the iterated commutator of the four commutators in (9).) Finally, by part (c) of Proposition 5.4, we conclude that G is not MIF either. $\square$

It is known that any acylindrically hyperbolic group contains non-cyclic free subgroups and hence does not satisfy any non-trivial identity [15]. It is also known that a non-cyclic torsion free hyperbolic group does not satisfy any non-trivial mixed identity [2]. We can generalizes these results as follows.

Corollary 5.10. *Let G be an acylindrically hyperbolic group. Then $G/K(G)$ is MIF. In particular, if $K(G) = \{1\}$, then G is MIF.*

Proof. We first note that the quotient group $G/K(G)$ is acylindrically hyperbolic by Lemma 2.9, so it suffices to deal with the case $K(G) = \{1\}$. In this case G is highly transitive by Theorem 1.2. Further, it is proved in [49, Corollary 1.5] that every infinite normal subgroup of G is acylindrically hyperbolic. $Alt(\mathbb{N})$ is not acylindrically hyperbolic since every acylindrically hyperbolic group contains infinite order elements by Theorem 2.6. Hence G cannot contain $Alt(\mathbb{N})$ as a normal subgroup. To complete the proof it remains to refer to Theorem 5.9. $\square$

A natural question arising from comparing Corollary 5.10 and Theorem 5.9 is whether every highly transitive MIF group is acylindrically hyperbolic. We will show that the answer is negative by making use of a result from [20].

Let A be a subgroup of a group G. Recall that for a subset $S \subseteq G$, the core of A with respect to S is defined by the formula $Core_S(A) = \bigcap_{s \in S} s^{-1}As$. Further, A is called *highly core free* in G if for every finite $F \subseteq G$, every $n \in \mathbb{N}$, and every subsets $S_1, \ldots, S_n \subseteq G$ such that

$$G = AF \cup S_1 \cup \ldots \cup S_n, \tag{10}$$

there exists $1 \leq k \leq n$ such that $Core_{S_k}(A) = \{1\}$. The following is Theorem 2.2 in [20].

Lemma 5.11. *Let H be an HNN-extension of a countable group G with associated subgroups A and B. Assume that A and B are highly core free in G. Then H is highly transitive.*

Corollary 5.12. *Let*
$$H = \langle a, b, t \mid b^{-1}ab = a^2, [a,t] = 1 \rangle.$$

Then H is highly transitive and MIF, but not acylindrically hyperbolic.

Proof. It is easy to see that the subgroup $A = \langle a \rangle$ is highly core free in $G = \langle a, b \mid a^b = a^2 \rangle$. Indeed for every finite $F \subseteq G$ and every decomposition (10), there exists at least one k such that the natural projection of S_k to $\langle b \rangle$ contains infinitely many positive powers of b. Obviously this implies $Core_{S_k}(A) = \{1\}$. Hence H is highly transitive by Lemma 5.11. Since G is torsion free so is H. Thus H does not contain a subgroup isomorphic to $Alt(\mathbb{N})$, so H is MIF by Theorem 5.9.

Observe that $A = \langle a \rangle$ is an s-normal subgroup of H; that is, $A \cap A^h$ is infinite for every $h \in H$. By [49, Corollary 1.5], any s-normal subgroup of an acylindrically hyperbolic group is acylindrically hyperbolic. This implies that H is not acylindrically hyperbolic. $\square$

Finally, we record the following explicit construction of finitely generated highly transitive subgroups satisfying condition (a) of Theorem 5.9.

Proposition 5.13. *For every finitely generated infinite group Q, there exists a finitely generated group G such that $FSym(\mathbb{N}) \triangleleft G \leq Sym(\mathbb{N})$ and $G/FSym(\mathbb{N}) \cong Q$. In particular, G is highly transitive.*

Proof. We identify $Sym(\mathbb{N})$ and $Sym(Q)$. Let $X = \{x_1, \ldots, x_k\}$ be a generating set of Q and let $\lambda \colon Q \to Sym(Q)$ be the embedding induced by the left regular action. Let a_i denote the transposition $(1, x_i) \in Sym(Q)$ and let $G = \langle \lambda(x_1), \ldots, \lambda(x_k), a_1, \ldots, a_k \rangle \leq Sym(Q)$.

Let us show that G contains $FSym(Q)$. Let $t_{g,h} = (g, h)$, $g, h \in Q$, be a transposition. Assume first that $g^{-1}h = x_i$ for some i. Then $t_{g,h} = \lambda(g)a_i\lambda(g^{-1})$. Similarly if $g^{-1}h = x_i^{-1}$, then $t_{g,h} = t_{h,g} = \lambda(h)a_i\lambda(h^{-1})$. In both cases $t_{g,h} \in G$. In the general case, there exist elements $g_0 = g, g_1, \ldots, g_n = h$, where $n = |g^{-1}h|_X$ and $g_{i+1} = g_i x_{j_i}^{\pm 1}$ for some $x_{i_j} \in X$. Then $t_{g,h} = t_{g_0,g_1} \cdots t_{g_{n-1},g_n}$. Thus $t_{g,h} \in G$ for every $g, h \in Q$ and therefore $FSym(Q) \leq G$.

Clearly $FSym(Q)$ is normal in G since it is normal in $Sym(Q)$. Note also that $\lambda(Q) \cap FSym(Q) = \{1\}$. Indeed every element of $\lambda(Q)$ has single (infinite) orbit since the action of Q on itself is the left regular one, while every element of $FSym(Q)$ moves only finitely many elements. Thus $G = FSym(Q) \rtimes \lambda(Q) \cong FSym(Q) \rtimes Q$. $\qquad\square$

Corollary 5.14. *There exist infinite finitely generated torsion groups admitting highly transitive faithful actions.*

Proof. Take a finitely generated infinite torsion group Q and apply the proposition. $\qquad\square$

5.3. Mixed identities in limits of acylindrically hyperbolic groups. Our next goal is to show how Corollary 5.10 can be used to construct examples of MIF groups which are very far from being acylindrically hyperbolic. We first recall the definition of the space of marked group presentations. It is a particular case of a more general construction due to Chabauty; for groups it was first studied by Grigorchuk [24].

Let F_k be the free group with basis $X = \{x_1, \ldots, x_k\}$ and let $\mathcal{G}_k$ denote the set of all normal subgroups of F_k. Given $M, N \triangleleft F_k$, let

$$
\mathrm{d}(M, N) = \begin{cases} \max\left\{ \frac{1}{|w|_X} \ \middle|\ w \in N \, \triangle \, M \right\}, & \text{if } M \neq N \\ 0, & \text{if } M = N. \end{cases}
$$

It is easy to see that $(\mathcal{G}_k, \mathrm{d})$ is a compact Hausdorff totally disconnected (ultra)metric space [24].

One can naturally identify $\mathcal{G}_k$ with the set of all *marked k-generated groups*, i.e., pairs $(G, (x_1, \ldots, x_k))$, where G is a group and $(x_1, \ldots, x_k)$ is a generating k-tuple of G. For this reason the space $\mathcal{G}_k$ with the metric defined above is called the *space of marked groups with k generators.*

Let $\mathcal{AH}_k^0$ denote the set of all marked k-generated acylindrically hyperbolic groups with trivial finite radical and let $\overline{\mathcal{AH}}_k^0$ denote its closure in $\mathcal{G}_k$. In the proposition below, the word 'generic' is used in the sense of Definition 5.7.

Proposition 5.15. *Let $\mathcal{C}$ be a subset of $\mathcal{G}_k$ for some $k \geq 2$ such that $\mathcal{C} \cap \mathcal{AH}_k^0$ is dense in $\mathcal{C}$. Then a generic presentation from $\mathcal{C}$ defines a MIF group.*

Proof. Let $F_{k+1} = F(x_1, \ldots, x_k) * \langle x \rangle = \{1, w_1, w_2, \ldots\}$. If G is a marked k-generated group (i.e., a quotient group of F_k), we denote by $\bar{w}_i$ the natural image of w_i in $G * \langle x \rangle$. Let $\mathcal{D}_i$ be the set of all marked k-generated groups G such that $\bar{w}_i = 1$ in $G * \langle x \rangle$ or $\bar{w}_i(g) \neq 1$ in G for some $g \in G$. It is easy to see that if a certain marked k-generated group G belongs to $\mathcal{D}_i$, then all marked k-generated groups from some open neighborhood of G belong to $\mathcal{D}_i$. Thus $\mathcal{D}_i$ is open. Further, let $\mathcal{D} = \bigcap_i \mathcal{D}_i$. Clearly $\mathcal{D}$ consists of MIF groups (see Remark 5.1). By Corollary 5.10, $\mathcal{AH}_k^0 \subseteq \mathcal{D}$. Thus $\mathcal{D}$ is an intersection of countably many open dense subsets of $\mathcal{C}$, i.e., it is comeagre in $\mathcal{C}$. $\qquad\square$

By Proposition 5.3, every countable MIF group is universally equivalent to an acylindrically hyperbolic group (namely, the free product $G * F_n$). The following corollary of Proposition 5.15 shows that universal equivalence cannot be replaced by elementary equivalence and even with $\forall\exists$-equivalence here.

Corollary 5.16. *There exist a finitely generated MIF group with exactly* 2 *conjugacy classes. In particular, there exists a MIF group that is not elementary equivalent (not even $\forall\exists$-equivalent) to an acylindrically hyperbolic group.*

Proof. Let $\mathcal{AH}_k^{tf}$ denote the set of all marked presentations of k-generated torsion free acylindrically hyperbolic groups. Clearly $\mathcal{AH}_k^{tf} \subseteq \mathcal{AH}_k^0$. It was proved in [32, Corollary 1.13] that a generic presentation in $\overline{\mathcal{AH}}_k^{tf}$ defines a group with 2 conjugacy classes. On the other hand, a generic presentation in $\overline{\mathcal{AH}}_k^{tf}$ defines a MIF group by Proposition 5.15. Since intersection of two comeagre sets is comeagre, a generic presentation in $\overline{\mathcal{AH}}_k^{tf}$ defines a group satisfying both properties. It remains to note that $\overline{\mathcal{AH}}_k^{tf}$ is compact being a closed subset of a compact space, and hence comeagre subsets of $\overline{\mathcal{AH}}_k^{tf}$ are non-empty by the Baire category theorem.

Finally we note that groups with exactly 2 conjugacy classes can be characterized by the $\forall\exists$-formula
$$\forall x \, \forall y \, \exists t \, (x = 1 \; \vee \; y = 1 \; \vee \; t^{-1}xt = y).$$

Note also that acylindrically hyperbolic groups have infinitely many conjugacy classes (and, moreover, exponential conjugacy growth, see [33]). This implies the second claim of the corollary. □

Our next result provides examples of MIF groups which are not highly transitive.

Corollary 5.17. *There exists MIF torsion groups of transitivity degree* 1.

Proof. Let $\mathcal{C}_k$ be the class of all marked k-generated hyperbolic groups H such that

(*) H is not virtually cyclic.

(**) For any $x, y \in H$, $[x^2, y] = 1$ implies $[x, y] = 1$.

Olshanskii showed that for every $k \geq 2$, the closure $\overline{\mathcal{C}}_k$ contains a dense subset consisting of torsion groups (this is an immediate consequence of [45, Theorems 3 and 4]). Let $F_k = \{1, w_1, w_2, \ldots\}$. For every $i \in \mathbb{N}$, let $\mathcal{T}_i$ denote the subset of $\mathcal{G}_k$ consisting of marked groups G such that w_i has finite order in G. Obviously $\mathcal{T}_i$ is open and the set $\mathcal{T} = \bigcap_i \mathcal{T}_i$ consists of torsion groups. Combining this with the Olshanskii's result and Proposition 5.15, we obtain that the set of MIF torsion groups is comeagre in $\overline{\mathcal{C}}_k$. In particular, there exists a torsion MIF marked k-generated group $T \in \overline{\mathcal{C}}_k$.

Notice that every marked group from $\overline{\mathcal{C}}_k$ satisfies (**). Indeed suppose that (**) fails in some marked group $G \in \overline{\mathcal{C}}_k$. Then it fails in any marked group in some open neighborhood of G (for the same elements x and y as in G). Hence (**) fails in some group from $\mathcal{C}_k$, a contradiction. In particular, T satisfies (**) and therefore every involution in T is central. Thus if T has an involution, it has $\mathrm{td}(T) = 1$ by Lemma 4.2 (b). If T does not have involutions, then every element of T has odd order and hence no subgroup of T can surject on $\mathbb{Z}_2$. On the other hand, in any 2-transitive permutation group acting on a set Ω, the quotient of the setwise stabilizer of any pair of elements of Ω modulo the pointwise stabilizer of the same elements is isomorphic $\mathbb{Z}_2$. Thus, in any case, we obtain $\mathrm{td}(T) = 1$. $\qquad\square$

6 Open questions

In this section we discuss some open questions about transitivity degree of countable groups. We begin with two problems mentioned in the introduction.

Question 6.1. *Does there exist infinite countable groups of transitivity degree k for every $k \in \mathbb{N}$?*

There are examples for $k = 1, 2, 3, \infty$. We discuss the cases $k = 2$ and $k = 3$ below. Starting from $k = 4$ the problem seems open.

It is well-known and easy to prove that for every field F, the natural action of the 1-dimensional affine group $Aff(1, F)$ on F is 2-transitive. Recall that $Aff(1, F)$ is the group of transformations of F^2 generated by the linear maps $x \mapsto x + a$ and $x \mapsto bx$ for $a \in F$ and $b \in F^*$. Note also that if F is infinite, then $Aff(1, F)$ does not admit any 3-transitive faithful action. Indeed, it is easy to see that $Aff(1, F) = F \rtimes F^*$. If $Aff(1, F)$ has a faithful 3-transitive action on a set Ω and $F \neq \mathbb{F}_2$, then the action of $F \lhd Aff(1, F)$ on Ω must be 2-transitive by the Cameron's theorem (Lemma 4.16). However this contradicts Lemma 4.2 (b). Thus $\mathrm{td}(Aff(1, F)) = 2$. Another example of groups of transitivity degree 2 is given in Example 5.16.

To construct a countable group G of transitivity degree 3, we slightly modify an example from [13] showing that part (b) of Lemma 4.16 does realize. Let V be the direct sum of countably many copies of $\mathbb{Z}_2$. We think of V as a vector space over $\mathbb{F}_2$ with basis $e_1, e_2, \ldots$. Let A be the group of automorphisms of V with finite support. Let $G = A \rtimes V$ be the corresponding split extension. The action of V on itself by translations and the action of A by automorphisms induces a faithful (affine) action of G on V. Let x, y, z be any distinct vectors of V. Using the translation action of V, we can shift this triple to $0, u, v$, where $u = y - x$, $v = z - x$. Note that $u \neq v$ and both vectors are non-zero; since the field of scalars is $\mathbb{F}_2$, u and v are linearly independent. Hence there is an automorphism of V with finite support, which maps the triple $0, u, v$ to $0, e_1, e_2$. This shows that $\mathrm{td}(G) \geq 3$. On the other hand, if $\mathrm{td}(G) = 4$, then the abelian group V must be 3-transitive by Lemma 4.16, which contradicts Lemma 4.2 (b).

Question 6.2. *Suppose that* $\mathrm{td}(G) = \infty$. *Is* G *highly transitive?*

In general, it is rather difficult to show that a certain group with $\mathrm{td}(G) = \infty$ *does not* admit a highly transitive action. To do this, one can try to find a property of groups which can hold in a k-transitive group for any k, but never holds for a highly transitive group. We wonder if satisfying a non-trivial mixed identity is such a property. In other words, we would like to know if every countable group with $\mathrm{td}(G) = \infty$ is MIF.

Question 6.3. *Compute transitivity degrees of some interesting particular groups. Concrete examples include* $PGL(2, F)$ *for a countably infinite field* F, *free Burnside groups, Thompson's group* F, *Baumslag-Solitar groups, etc.*

It is well-known that the action of $PGL(2, F)$ on the corresponding projective space is 3-transitive. Thus $\mathrm{td}(PGL(2, F)) \geq 3$. The action of $PGL(2, F)$ on the projective space is not 4-transitive if F is infinite, but we cannot exclude the possibility that there exists another faithful action which is 4-transitive. Thus we do not know what $\mathrm{td}(PGL(2, F))$ is. We do not even know whether $PGL(2, F)$ is highly transitive or not. Note that $PGL(2, F)$ is not acylindrically hyperbolic. Indeed it is either finite or simple by the Jordan-Dixon theorem, while acylindrically hyperbolic groups are never simple [15].

It is easy to see that transitivity degree of any solvable Baumslag-Solitar group is 1 (e.g., by Corollary 4.6). The same holds for $\langle a, t \mid t^{-1}a^m t = a^{\pm m} \rangle$; indeed in this case the conjugacy class of a^m is finite and we can apply Lemma 4.2 (b). The first interesting case is $BS(2, 3) = \langle a, t \mid t^{-1}a^2 t = a^3 \rangle$. We do not even know if $BS(2, 3)$ is highly transitive.

Alexander Olshanskii (private communication) noted that transitivity degree of a free Burnside group of exponent n is finite and does not exceed the maximal number k such that n is divisible by all positive integers $p \leq k$. Indeed if G acts on $\mathbb{N}$ faithfully and k-transitively, then there is a subgroup $H \leq G$ (namely, the setwise stabilizer of $\{1, \ldots, k\}$) that admits an epimorphism onto the symmetric group S_k. Hence for every positive integer $p \leq k$, H must contain an element whose order is divisible by p. In particular, transitivity degree of a free Burnside group of odd exponent is 1. It is quite possible that this is also true for all infinite free Burnside groups of sufficiently large exponent.

Question 6.4. *Is being highly transitive a weakly geometric property? That is, if* G *is quasi-isometric to a highly transitive group, is it commensurable to such a group?*

Note that being highly transitive passes to subgroups of finite index but not to finite extensions. So if we replace "commensurable" with "isomorphic", the answer is negative.

Question 6.5. *Does there exist a countable amenable highly transitive subgroup* $A \leq Sym(\mathbb{N})$ *such that* $A \cap Alt(\mathbb{N}) = \{1\}$?

If we drop the assumption $A \cap Alt(\mathbb{N}) = \{1\}$, then there are even finitely presented examples.

Example 6.6. Fix some $n \in \mathbb{N}$ and let Ω_n be the disjoint union of n copies $\mathbb{N}_1, \ldots, \mathbb{N}_n$ of $\mathbb{N}$. Let H_n be the group of all permutations s of Ω_n such that s acts as a translation along each N_i outside a finite set. These groups, commonly called Houghton groups, were introduced in [30]. It is known that H_n if finitely generated for $n \geq 2$ and finitely presented for $n \geq 3$. Further it is not hard to show that for $n \geq 2$, the derived subgroup of H_n coincides with $FSym(\Omega_n)$. Thus H_n is (locally finite)-by-abelian (in particular, it is elementary amenable) and highly transitive. For details, see [12].

As the first step towards answering Question 6.5, we propose the following.

Question 6.7. *Construct countable (or, better, finitely generated) MIF amenable groups.*

It is likely that lacunary hyperbolic amenable groups constructed in [46] are MIF. Recall that a group G is *lacunary hyperbolic* if it is finitely generated and at least one asymptotic cone of G is an $\mathbb{R}$-tree. In fact, we conjecture the following.

Conjecture 6.8. *Every lacunary hyperbolic group that is not virtually cyclic and has no finite normal subgroups is MIF.*

This should follow from the following fact: Every non-trivial mixed identity $w \in G * \langle x \rangle$ in a δ-hyperbolic group G with trivial finite radical can be violated by an element $g \in G$ of length at most $C(\delta + 1)(s(w) + 1)$, where $s(w)$ is the sum of lengths (in G) of elements $a_1, \ldots, a_k \in G$ involved in the normal form of w. It seems plausible that this result can be proved by taking g to be a high power of an infinite order element $h \in G$ such that no a_i belongs to $E(h)$ (i.e., to the maximal virtually cyclic subgroup of G containing h). However verifying details of this proof is outside of the scope of our paper.

Lacunary hyperbolic groups can be thought of as infinitely presented analogues of hyperbolic groups. We note, however, that Chainikov's theorem about highly transitive actions of hyperbolic groups does not extend to lacunary hyperbolic ones.

Example 6.9. There exists a non-virtually cyclic torsion free lacunary hyperbolic group G such that $td(G) = 1$. Indeed let G be a non-cyclic torsion free lacunary hyperbolic group such that every proper subgroup of G is cyclic. Such groups were constructed in [46]. Assume that G acts 2-transitively on a set X and let H be the stabilizer of some $x \in X$. Then H is infinite cyclic. However Mazurov [38] proved that there are no 2-transitive permutation groups with infinite cyclic point stabilizers.

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